

# Extended-valued topical and anti-topical functions on semimodules

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## Abstract

In the papers [13] and [14] we have studied functions  $f : X \rightarrow \mathcal{K}$  defined on a  $b$ -complete semimodule  $X$  over an idempotent  $b$ -complete semifield  $\mathcal{K} = (\mathcal{K}, \oplus, \otimes)$ , with values in  $\mathcal{K}$ , where  $\mathcal{K}$  may (or may not) contain a greatest element  $\sup \mathcal{K}$ , and the residuation  $x/y$  is not defined for  $x \in X$  and  $y = \inf X$ . In the present paper we assume that  $\mathcal{K}$  has no greatest element (equivalently,  $\mathcal{K}$  is not a singleton and not the two-element Boolean semifield), then adjoin to  $\mathcal{K}$  an outside “greatest element”  $\top = \sup \mathcal{K}$  and extend the operations  $\oplus$  and  $\otimes$  from  $\mathcal{K}$  to  $\overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$ , so as to obtain a meaning also for  $x/\inf X$ , for any  $x \in X$ , and study the functions  $f : X \rightarrow \overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$ . In fact we introduce two different extensions of the product  $\otimes$  from  $\mathcal{K}$  to  $\overline{\mathcal{K}}$ , denoted by  $\otimes$  and  $\dot{\otimes}$  respectively, and use them to give characterizations of topical (i.e. increasing homogeneous, defined with the aid of  $\otimes$ ) and anti-topical (i.e. decreasing anti-homogeneous, defined with the aid of  $\dot{\otimes}$ ) functions  $f : X \rightarrow \overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$  with the aid of some inequalities. Next we introduce and study for functions  $f : X \rightarrow \overline{\mathcal{K}} = \mathcal{K} \cup \{\top\}$  their conjugates and biconjugates of Fenchel-Moreau type with respect to the coupling functions  $\varphi(x, y) = x/y, \forall x \in X, \forall y \in X$ , and  $\psi(x, (y, d)) := \inf\{x/y, d\}, \forall x \in X, \forall y \in X, \forall d \in \overline{\mathcal{K}}$ , and use them for obtaining characterizations of topical and anti-topical functions. In the subsequent sections we consider for the coupling functions  $\varphi$  and  $\psi$  some concepts that have been studied in [9] and [12] for the so-called “additive min-type coupling functions”  $\pi_\mu : R_{\max}^n \times R_{\max}^n \rightarrow R_{\max}$  and  $\pi_\mu : A^n \times A^n \rightarrow A$  respectively, where  $A$  is a conditionally complete lattice ordered group and  $\pi_\mu(x, y) := \inf_{1 \leq i \leq n} (x_i + y_i), \forall x, y \in R_{\max}^n$  (or  $A^n$ ). Thus, we study the polars of a set  $G \subseteq X$  for the coupling functions  $\varphi$  and  $\psi$ , and we consider the support set of a function  $f : X \rightarrow \overline{\mathcal{K}}$  with respect to the set  $\tilde{\mathcal{T}}$  of all “elementary topical functions”  $\tilde{t}_y(x) := x/y, \forall x \in X, \forall y \in X \setminus \{\inf X\}$  and

two concepts of support set of  $f : X \rightarrow \overline{\mathcal{K}}$  at a point  $x_0 \in X$ . The main differences between the properties of the conjugations with respect to the coupling functions  $\varphi, \psi$  and  $\pi_\mu$  and between the properties of the polars of a set  $G$  with respect to the coupling functions  $\varphi, \psi$  and  $\pi_\mu$  are caused by the fact that while  $\pi_\mu$  is symmetric, with values only in  $R_{\max}$  (resp.  $A$ ),  $\varphi$  and  $\psi$  are not symmetric and take values also outside  $R_{\max}$  (resp.  $A$ ).

*Key Words:* Semifield, semimodule,  $b$ -complete, extended product, extended-valued function, topical function, anti-topical function, elementary topical function, Fenchel-Moreau conjugate, biconjugate, support function, polar, bipolar, support set, downward set, subdifferential

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## 1 Introduction

In the previous papers [13] and [14], attempting to contribute to the construction of a theory of functional analysis and convex analysis in semimodules over semifields, we have studied topical functions  $f : X \rightarrow \mathcal{K}$  and related classes of functions, where  $X$  is a  $b$ -complete semimodule over an idempotent  $b$ -complete semifield  $\mathcal{K}$ . We recall that  $f : X \rightarrow \mathcal{K}$  is called *topical* if it is *increasing* (i.e., the relations  $x', x'' \in X, x' \leq x''$  imply  $f(x') \leq f(x'')$ , where  $\leq$  denotes the canonical order on  $\mathcal{K}$ , respectively on  $X$ , defined by  $\lambda \leq \mu \Leftrightarrow \lambda \oplus \mu = \mu, \forall \lambda \in \mathcal{K}, \forall \mu \in \mathcal{K}$ , respectively by  $x \leq y \Leftrightarrow x \oplus y = y, \forall x \in X, \forall y \in X$ ), and *homogeneous* (i.e.,  $f(\lambda x) = \lambda f(x)$  for all  $x \in X, \lambda \in \mathcal{K}$ , where  $\lambda x := \lambda \otimes x, \lambda f(x) = \lambda \otimes f(x)$ ; the fact that we use the same notations for addition  $\oplus$  both in  $\mathcal{K}$  and in  $X$  and for multiplication  $\otimes$  both in  $\mathcal{K}$  and in  $\mathcal{K} \times X$  will lead to no confusion). These definitions will be used also when  $\mathcal{K}$  is replaced by  $R = ((-\infty, +\infty), \oplus = \max, \otimes = +)$  although it is not a semiring, and  $X$  is replaced by  $R^n$ . Let us also recall that a semiring  $\mathcal{K}$ , or a semimodule  $X$  (over a semiring  $\mathcal{K}$ ) is called  *$b$ -complete*, if it is closed under the sum  $\oplus$  of any subset (order-) bounded from above and the multiplication  $\otimes$  distributes over such sums.

As in [13] and [14], we shall make the following *basic assumptions*:

(A0')  $\mathcal{K} = (\mathcal{K}, \oplus, \otimes)$  is a  *$b$ -complete semifield* (i.e., a  $b$ -complete semiring in which every  $\mu \in \mathcal{K} \setminus \{\varepsilon\}$  is invertible for the multiplication  $\otimes$ , where  $\varepsilon$  denotes the neutral element of  $(\mathcal{K}, \oplus)$ ), *with idempotent addition*  $\oplus$  (i.e. such that  $\lambda \oplus \lambda = \lambda$  for all  $\lambda \in \mathcal{K}$ ), and the supremum of each (order-) bounded from above subset of  $\mathcal{K}$  belongs to  $\mathcal{K}$ ; also,  $X$  is a  *$b$ -complete semimodule* over  $\mathcal{K}$ . In the sequel we shall omit the word “idempotent”; this will lead to no confusion.

(A1) For all elements  $x \in X$  and  $y \in X \setminus \{\inf X\}$  the set  $\{\lambda \in \mathcal{K} \mid \lambda y \leq x\}$  is (order-) bounded from above, where  $\leq$  denotes the canonical order on  $\mathcal{K}$ , respectively on  $X$ ,

**Remark 1** a) Refining a remark of [4, p.415], it follows that  $\mathcal{K}$  has no greatest element  $\sup \mathcal{K}$ , unless  $\mathcal{K} = \{\varepsilon\}$  or  $\mathcal{K} = \{\varepsilon, e\}$ , where  $\varepsilon$  and  $e$  denote the neutral elements of  $(\mathcal{K}, \oplus)$  and  $(\mathcal{K}, \otimes)$  respectively. For, let  $\mathcal{K} \neq \{\varepsilon\}$  and  $\mathcal{K} \neq \{\varepsilon, e\}$  and

assume, a contrario, that  $\sup \mathcal{K} \in \mathcal{K}$ . Then

$$\sup \mathcal{K} = (\sup \mathcal{K}) \otimes e \leq (\sup \mathcal{K}) \otimes (\sup \mathcal{K}) \leq \sup \mathcal{K},$$

whence  $(\sup \mathcal{K}) \otimes (\sup \mathcal{K}) = \sup \mathcal{K}$ ; since  $(\sup \mathcal{K})^{-1} \neq \varepsilon$ , multiplying both sides of this equality with  $(\sup \mathcal{K})^{-1}$  we obtain  $\sup \mathcal{K} = e$ , so  $\lambda \leq e$  for all  $\lambda \in \mathcal{K}$ . Since  $\mathcal{K}$  is a semifield, replacing here each  $\lambda \in \mathcal{K} \setminus \{\varepsilon\}$  by  $\lambda^{-1}$  it follows that  $\lambda \geq e$  and hence  $\lambda = e$ , for all  $\lambda \in \mathcal{K} \setminus \{\varepsilon\}$ . Consequently,  $\mathcal{K} = \{\varepsilon, e\}$ , in contradiction with our assumption. The converse statement is obvious: if  $\mathcal{K} = \{\varepsilon, e\}$ , then  $\sup \mathcal{K} = e \in \mathcal{K}$ .

*In the sequel, without any special mention, we shall assume that  $\mathcal{K}$  has no greatest element*, since for  $\mathcal{K} = \{\varepsilon\}$  the statements are trivial and for  $\mathcal{K} = \{\varepsilon, e\}$  the subsequent results remain valid with similar but simpler proofs.

b)  $\mathcal{K}$  is commutative, by (A0') and Iwasawa's theorem (see e.g. [3]).

An important example of a pair  $(X, \mathcal{K})$  satisfying (A0') and (A1) is obtained by taking

$$X = R_{\max}^n := ((R \cup \{-\infty\})^n, \oplus := \max, \otimes := +) \quad (1)$$

(with  $\max$  and  $+$  understood componentwise), and  $\mathcal{K} := R_{\max}^1$ . The results of [9] on  $R^n := (R, \oplus := \max, \otimes := +)^n$  can and will be expressed in the sequel as results on  $R_{\max}^n$  replacing  $R$  by  $R \cup \{-\infty\}$  endowed with the usual operations  $\max$  and  $+$ . Also, as has been observed in [9], many results on  $R^n$  remain valid, essentially with the same proofs, for  $X = R_{\max}^I$ , the set of all bounded vectors  $x = (x_i)_{i \in I}$  where  $I$  is an arbitrary index set and  $x_i \in R, \forall i \in I, \sup_{i \in I} |x_i| < +\infty$ , endowed with the componentwise semimodule operations  $x' \oplus x'' := (\max(x'_i, x''_i))_{i \in I}$ ,  $\lambda x = (\lambda x_i)_{i \in I}$  and the componentwise order relation  $x' \leq x'' \Leftrightarrow x'_i \leq x''_i, \forall i \in I$ .

One of the main tools in [13] and [14] has been residuation. We recall that by (A0') and (A1), for each  $y \in X \setminus \{\inf X\}$  (hence such that the set  $\{\lambda \in \mathcal{K} | \lambda y \leq x\}$  is (order-) bounded from above, by (A1)) there exists the *residuation operation* / defined by

$$x/y := \max\{\lambda \in \mathcal{K} | \lambda y \leq x\}, \quad \forall x \in X, \forall y \in X \setminus \{\inf X\}, \quad (2)$$

where  $\max$  denotes a supremum that is attained, and it has, among others, the following properties (see e.g. [4]):

$$(x/y)y \leq x, \quad \forall x \in X, \forall y \in X \setminus \{\inf X\}, \quad (3)$$

$$y/y = e, \quad \forall y \in X \setminus \{\inf X\}, \quad (4)$$

$$x/(\mu y) = \mu^{-1}(x/y), \quad \forall x \in X, \forall \mu \in \mathcal{K} \setminus \{\varepsilon\}, \forall y \in X \setminus \{\inf X\}. \quad (5)$$

In [13] and [14] we have considered only functions  $f : X \rightarrow \mathcal{K}$ , where  $(X, \mathcal{K})$  is a pair satisfying (A0') and (A1). Therefore by (A1), under the assumption of Remark 1a) above  $x/\inf X$ , i.e. the residuation  $x/y$  of (2) for  $x \in X$  and  $y = \inf X$ , is not defined. In the present paper we shall adjoin to  $\mathcal{K}$  an outside "greatest element"  $\sup \mathcal{K}$  which we shall denote by  $\top$ , and extending in a suitable

way the operations  $\oplus$  and  $\otimes$  from  $\mathcal{K}$  to  $\overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$ , we shall then study functions  $f : X \rightarrow \overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$ , that one may call “extended-valued functions”; for example, we shall study topical (i.e. increasing homogeneous) and anti-topical (i.e. decreasing anti-homogeneous) functions defined on a semimodule  $X$  over  $\mathcal{K}$  with values in  $\overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$ . Naturally, the extension of the sum operation to sets  $M \subseteq \overline{\mathcal{K}}$ , which we shall denote again by  $\oplus$ , must be  $\oplus M := \sup M$  (in  $\mathcal{K}$ ) if  $M$  is bounded from above in  $\mathcal{K}$  and  $\oplus M := \top = \sup \mathcal{K}$  if  $M$  is not bounded from above in  $\mathcal{K}$ . Also, generalizing the lower addition  $\dot{+}$  and upper

addition  $\dot{+}$  on  $\overline{R}$ , of Moreau (see e.g. [8]), we shall give two different extensions of the product  $\otimes$  from  $\mathcal{K}$  to  $\overline{\mathcal{K}}$ , denoted respectively by  $\otimes$  (which will cause no confusion) and  $\dot{\otimes}$ , that are dual to each other in a certain sense, so as to obtain a meaning also for  $x/\inf X$ , with any  $x \in X$ , and in particular for  $\inf X/\inf X$ . Let us mention that in the particular case of the pair  $(X, \mathcal{K}) = (R_{\max}^n, R_{\max}^1)$  such extended products and a table of the values of the residuations  $x/y$  for all  $x, y \in R_{\max}^1 \cup \{+\infty\}$  have been given in [1, Table 1].

First, in Section 2 we shall introduce and study the extended addition  $\oplus$  and the extended products  $\otimes$  and  $\dot{\otimes}$  in  $\overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$  and in Section 3 we shall use them to give characterizations of topical and anti-topical functions  $f : X \rightarrow \overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$  with the aid of some inequalities. For the case of topical functions these characterizations extend some results of [14].

Another domain where topical and anti-topical functions  $f : X \rightarrow \overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$  play an important role is that of conjugate functions of Fenchel-Moreau type with respect to coupling functions  $\pi : X \times X \rightarrow \overline{\mathcal{K}}$ , defined by  $f^{c(\pi)}(y) := \sup_{x \in X} f(x)^{-1} \pi(x, y)$ ,  $\forall y \in X$ . In Section 4 the extended products  $\otimes$  and  $\dot{\otimes}$  will permit us to introduce for functions  $f : X \rightarrow \overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$  their conjugates and biconjugates with respect to the coupling functions  $\varphi(x, y) = x/y$ ,  $\forall x \in X, \forall y \in X$ , and  $\psi(x, (y, d)) := \inf\{x/y, d\}$ ,  $\forall x \in X, \forall y \in X, \forall d \in \overline{\mathcal{K}}$ , and to use them for the study of topical and anti-topical functions. We shall also consider the “lower conjugates” of  $f$  with respect to these coupling functions, defined with the aid of the product  $\dot{\otimes}$ , that are useful for the study of biconjugates. The main differences between the properties of the conjugations with respect to the coupling functions  $\varphi, \psi$  and the so-called “additive min-type coupling functions”  $\pi_\mu : R_{\max}^n \times R_{\max}^n \rightarrow R_{\max}$ , resp.  $\pi_\mu : A^n \times A^n \rightarrow A$ , where  $A$  is a conditionally complete lattice ordered group, studied previously e.g. in [9], respectively [12], defined by  $\pi_\mu(x, y) := \inf_{1 \leq i \leq n} (x_i \otimes y_i)$ ,  $\forall x = (x_i) \in R_{\max}^n$  (resp.  $A^n$ ),  $\forall y = (y_i) \in R_{\max}^n$  (resp.  $A^n$ ), are caused by the fact that while  $\pi_\mu$  is “symmetric” (i.e.,  $\pi_\mu(x, y) = \pi_\mu(y, x)$ ,  $\forall x \in X, \forall y \in X$ ) and takes values only in  $R_{\max}$  (resp.  $A$ ),  $\varphi$  and  $\psi$  are not symmetric and take also the value  $+\infty$  (resp.  $\top$ ); for example, since  $\pi_\mu(x, y)$  is topical both in  $x$  and in  $y$ , while  $\varphi(x, y)$  is topical as a function of  $x$  and anti-topical as a function of  $y$ , it follows that while  $f^{c(\pi_\mu)} : R_{\max}^n \rightarrow \overline{R}$  is always a topical function, the conjugate function  $f^{c(\varphi)} : X \rightarrow \overline{\mathcal{K}}$  is always anti-topical.

In the subsequent sections we shall consider for the coupling functions  $\varphi$  and  $\psi$  some concepts that have been studied previously for the additive min-type coupling function  $\pi_\mu : R_{\max}^n \times R_{\max}^n \rightarrow R_{\max}$  and  $\pi_\mu : A^n \times A^n \rightarrow A$ , in [9] and

[12] respectively. Thus, in Section 5 we shall study the polars of a set  $G \subseteq X$  for the coupling functions  $\varphi$  and  $\psi$ , and in Section 6 we shall consider the support set of a function  $f : X \rightarrow \overline{\mathcal{K}}$  with respect to the set  $\widetilde{\mathcal{T}}$  of all “elementary topical functions”  $\widetilde{t}_y(x) := x/y, \forall x \in X, \forall y \in X \setminus \{\inf X\}$  and two concepts of support set of  $f : X \rightarrow \overline{\mathcal{K}}$  at a point  $x_0 \in X$ . While for functions  $f : X \rightarrow \overline{\mathcal{K}}$  the theory of conjugations  $f \rightarrow f^{c(\varphi)}$  is of interest, we shall show that for subsets  $G$  of  $X$  the theory of polarities  $G \rightarrow G^{o(\varphi)}$  permits to obtain some relevant results. Similarly to the case of conjugates of functions, the main differences between the properties of the polars of a set  $G$  with respect to the coupling functions  $\varphi, \psi$  and the additive min-type coupling function  $\pi_\mu : R_{\max}^n \times R_{\max}^n \rightarrow R_{\max}$ , are caused by the fact that while  $\pi_\mu$  is symmetric and takes only values in  $R_{\max}$ ,  $\varphi$  and  $\psi$  are not symmetric and take also the value  $+\infty$  (resp.  $\top$ ).

## 2 Extension of $\mathcal{K}$ to $\overline{\mathcal{K}} = \mathcal{K} \cup \{\top\}$

**Definition 2** Let  $\mathcal{K} = (\mathcal{K}, \oplus, \otimes)$  be a  $b$ -complete semifield that has no greatest element. We shall adjoin to  $\mathcal{K}$  an outside element, which we shall denote by  $\top$ , and we shall extend the canonical order  $\leq$  and the addition  $\oplus$  from  $\mathcal{K}$  to an (canonical) order  $\leq$  and an addition  $\oplus$  on  $\overline{\mathcal{K}} = \mathcal{K} \cup \{\top\}$  by

$$\varepsilon \leq \alpha \leq \top, \quad \forall \alpha \in \overline{\mathcal{K}}, \quad (6)$$

$$\alpha \oplus \top = \top \oplus \alpha = \top, \quad \forall \alpha \in \overline{\mathcal{K}}; \quad (7)$$

hence the equivalence  $\alpha \leq \beta \Leftrightarrow \alpha \oplus \beta = \beta$  remains valid for all  $\alpha, \beta \in \overline{\mathcal{K}}$ . Furthermore, we shall extend the multiplication  $\otimes$  from  $\mathcal{K}$  to  $\overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$  to two multiplications  $\dot{\otimes}$  and  $\otimes$  by the following rules:

$$\alpha \dot{\otimes} \beta = \alpha \otimes \beta, \quad \forall \alpha \in \mathcal{K}, \forall \beta \in \mathcal{K}, \quad (8)$$

$$\alpha \dot{\otimes} \top = \top \dot{\otimes} \alpha = \top, \quad \forall \alpha \in \overline{\mathcal{K}}, \quad (9)$$

$$\alpha \otimes \top = \top \otimes \alpha = \top, \quad \forall \alpha \in \overline{\mathcal{K}} \setminus \{\varepsilon\}, \quad (10)$$

$$\alpha \otimes \varepsilon = \varepsilon \otimes \alpha = \varepsilon, \quad \forall \alpha \in \overline{\mathcal{K}}. \quad (11)$$

We shall denote the extended product  $\otimes$  also by concatenation.

For the inverses in  $\overline{\mathcal{K}}$  with respect to  $\otimes$  we shall make the convention

$$\varepsilon^{-1} := \top, \quad \top^{-1} := \varepsilon, \quad (12)$$

whence, by the above,

$$\varepsilon^{-1} \varepsilon = \top \varepsilon = \varepsilon \neq e, \quad \varepsilon^{-1} \dot{\otimes} \varepsilon = \top \dot{\otimes} \varepsilon = \top \neq e, \quad (13)$$

$$\top^{-1} \top = \varepsilon \top = \varepsilon \neq e, \quad \top^{-1} \dot{\otimes} \top = \varepsilon \dot{\otimes} \top = \top \neq e. \quad (14)$$

We shall call the set  $\overline{\mathcal{K}} = \mathcal{K} \cup \{\top\}$  endowed with the operations  $\oplus, \otimes$  and  $\dot{\otimes}$  the *minimal enlargement of  $\mathcal{K}$* .

**Remark 3** a) The product  $\dot{\otimes}$  on  $\overline{\mathcal{K}}$  is associative, i.e. we have

$$(\alpha \dot{\otimes} \beta) \dot{\otimes} \gamma = \alpha \dot{\otimes} (\beta \dot{\otimes} \gamma), \quad \forall \alpha, \beta, \gamma \in \overline{\mathcal{K}}. \quad (15)$$

Indeed, if  $\top$  occurs as a term in one side of (15), then that side must be equal to  $\top$  (by (9)) and hence one of the terms of the other side of (15), too, must be equal to  $\top$  (since if  $\lambda, \mu \in \mathcal{K}$  then  $\lambda \dot{\otimes} \mu = \lambda \otimes \mu \in \mathcal{K}$  by (8), so  $\lambda \dot{\otimes} \mu \neq \top$ ). On the other hand, if  $\top$  does not occur in any one of the terms of (15), then (15) holds by the usual associativity of  $\otimes$  on  $\mathcal{K}$ .

b) By the definition of  $\otimes$  (for  $\alpha \in \mathcal{K}$ ) and by (10) (for  $\alpha = e$ ), we have

$$\alpha \otimes e = e \otimes \alpha = \alpha, \quad \forall \alpha \in \overline{\mathcal{K}}; \quad (16)$$

furthermore, by (8) and (9) we have

$$\alpha \dot{\otimes} e = e \dot{\otimes} \alpha = \alpha, \quad \forall \alpha \in \overline{\mathcal{K}}, \quad (17)$$

i.e.,  $e$  is the unit element of  $\overline{\mathcal{K}}$  for both products  $\otimes$  and  $\dot{\otimes}$ .

c)  $\varepsilon^{-1}$  and  $\top^{-1}$  are called “inverses” only by abuse of language, as shown by (13) and (14).

d) We shall see that with the above definition, the notions and results of [13, 14] on functions  $f : X \rightarrow \mathcal{K}$ , where  $X$  is a semimodule over  $\mathcal{K}$ , admit extensions in the above sense to functions  $f : X \rightarrow \overline{\mathcal{K}}$ , for the extended product  $\otimes$  of (10), (11). Therefore in the sequel whenever we shall refer to a result of [14] or [13], we shall understand, without any special mention, its extension (using the above conventions) to functions  $f : X \rightarrow \overline{\mathcal{K}}$ , for the extended product  $\otimes$  on  $\overline{\mathcal{K}}$ .

e) Note that *a priori* the extended products (actions)  $\lambda \otimes x$  and  $\lambda \dot{\otimes} x$ , where  $\lambda \in \overline{\mathcal{K}}$  and  $x \in X$ , need not be defined, except that by the definition of a semimodule  $X$  over  $\mathcal{K}$ , we have

$$\lambda \inf X = \lambda \otimes \inf X := \inf X, \quad \forall \lambda \in \mathcal{K}. \quad (18)$$

**Definition 4** We extend formula (18) to  $\lambda = \top$  by defining

$$\top \inf X = \top \otimes \inf X := \inf X, \quad (19)$$

and we define

$$\lambda \dot{\otimes} \inf X := \lambda \otimes \inf X = \inf X, \quad \forall \lambda \in \overline{\mathcal{K}}. \quad (20)$$

**Remark 5** If one would define in any way the products  $\top x$ , where  $x \in X$ , then for each “homogeneous” function  $f : X \rightarrow \overline{\mathcal{K}}$ , in the “extended” sense  $f(\lambda x) = \lambda f(x), \forall x \in X, \forall \lambda \in \overline{\mathcal{K}}$ , we would necessarily have

$$f(\top x) = \begin{cases} \top f(x) = \top & \text{if } f(x) \neq \varepsilon \\ \top f(x) = \varepsilon & \text{if } f(x) = \varepsilon, \end{cases}$$

so  $f(\top x)$  could have only two values, namely either  $\top$  or  $\varepsilon$ .

**Definition 6** Using (18), we define the *extended residuation* in the semimodule  $X$  over  $\mathcal{K}$  by (2) and

$$x/\inf X := \sup\{\lambda \in \mathcal{K} \mid \lambda \inf X \leq x\} = \sup_{\lambda \in \mathcal{K}} \lambda = \top, \quad \forall x \in X, \quad (21)$$

$$\inf X/x := \sup\{\lambda \in \mathcal{K} \mid \lambda x \leq \inf X\} = \begin{cases} \varepsilon & \text{if } x \neq \inf X \\ \sup_{\lambda \in \mathcal{K}} \lambda = \top & \text{if } x = \inf X. \end{cases} \quad (22)$$

**Remark 7** a) In (21) we need to take sup instead of max, since the set  $\{\lambda \in \mathcal{K} \mid \lambda \inf X \leq x\}$  is not bounded in  $\mathcal{K}$  for any  $x \in X$  and by Remark 1a)  $\top \notin \mathcal{K}$ .

b) For  $x = \inf X$ , from (21) and/or (22) it follows that

$$\inf X/\inf X = \top. \quad (23)$$

We shall use the following extension of formula (5):

**Lemma 8** *We have*

$$x/(\mu y) = \mu^{-1} \dot{\otimes} (x/y) \quad \forall x, y \in X, \mu \in \mathcal{K}. \quad (24)$$

**Proof.** For  $y \in X \setminus \{\inf X\}$  and  $\mu \in \mathcal{K} \setminus \{\varepsilon\}$ , (24) reduces to (5).

For  $y \in X \setminus \{\inf X\}$  and  $\mu = \varepsilon$  we have  $x/(\varepsilon y) = x/\inf X = \top$  and  $\varepsilon^{-1} \dot{\otimes} (x/y) = \top \dot{\otimes} (x/y) = \top$ .

For  $y = \inf X$  and each  $\mu \in \mathcal{K}$  we have  $x/(\mu \inf X) = x/\inf X = \top, \forall x \in X$ , and  $\mu^{-1} \dot{\otimes} (x/\inf X) = \mu^{-1} \dot{\otimes} \top = \top, \forall x \in X$ .  $\square$

**Lemma 9** *For any  $x, y \in X$  such that either  $x \neq \inf X$  or  $y \neq \inf X$  we have*

$$(x/y)^{-1} = y/x. \quad (25)$$

**Proof.** If  $x, y \in X \setminus \{\inf X\}$ , then

$$\begin{aligned} (x/y)^{-1} &= (\max\{\lambda \in \mathcal{K} \mid \lambda y \leq x\})^{-1} \\ &= \min\{\lambda^{-1} \in \mathcal{K} \mid \lambda y \leq x\} \\ &= \min\{\lambda^{-1} \in \mathcal{K} \mid \lambda \leq xy^{-1}\} \\ &= (xy^{-1})^{-1} = yx^{-1} \\ &= \max\{\mu \in \mathcal{K} \mid \mu \leq yx^{-1}\} \\ &= \max\{\mu \in \mathcal{K} \mid \mu x \leq y\} = y/x. \end{aligned}$$

On the other hand, if either  $x \in X$  and  $y = \inf X$ , or  $x = \inf X$  and  $y \in X$ , then (25) follows from (21) and (22).  $\square$

**Remark 10** When  $x = y = \inf X$ , formula (25) is not valid, since  $x/y = y/x = \top$  and  $(x/y)^{-1} = (y/x)^{-1} = \varepsilon$ .

In the sequel the following properties of equivalence of some inequalities involving the extended products  $\otimes, \dot{\otimes}$  on  $\overline{\mathcal{K}}$  will be useful:

**Lemma 11** For all  $\lambda, \mu, \beta \in \overline{\mathcal{K}}$ :

A) The inequality

$$\lambda\mu \leq \beta \quad (26)$$

is equivalent to

$$\beta^{-1}\mu \leq \lambda^{-1}. \quad (27)$$

B) The inequality

$$\lambda \dot{\otimes} \mu \geq \beta \quad (28)$$

is equivalent to

$$\beta^{-1} \dot{\otimes} \mu \geq \lambda^{-1}. \quad (29)$$

**Proof.** A) The inequalities (26) and (27) are equivalent if  $\lambda, \mu, \beta \in \mathcal{K} \setminus \{\varepsilon\}$  (indeed, this follows immediately from the fact that  $\alpha\alpha^{-1} = e$  for any  $\alpha \in \mathcal{K} \setminus \{\varepsilon\}$ ). Thus it remains to consider the cases when one of  $\lambda, \mu$  or  $\beta$  is  $\varepsilon$  or  $\top$ .

Case (I):  $\lambda = \varepsilon$ . Then (26) means that  $\varepsilon = \varepsilon\mu \leq \beta$ , which is true for all  $\mu, \beta$ , and (27) means that  $\beta^{-1}\mu \leq \lambda^{-1} = \top$ , which is also true for all  $\mu, \beta$ . Hence (26)  $\Leftrightarrow$  (27).

Case (IIa):  $\lambda = \top$  and  $\mu = \varepsilon$ . Then (26) means that  $\varepsilon = \top\varepsilon \leq \beta$ , which is true for all  $\beta$ , and (27) means that  $\varepsilon = \beta^{-1}\mu \leq \lambda^{-1}$ , which is also true for all  $\beta$ . Hence (26)  $\Leftrightarrow$  (27).

Case (IIb):  $\lambda = \top$  and  $\mu \neq \varepsilon$ . Then (26) means that  $\top = \top\mu \leq \beta$ , which implies that  $\beta = \top$ , whence  $\beta^{-1}\mu = \varepsilon\mu \leq \lambda^{-1}$ , so (26) implies (27). In the reverse direction, (27) means that  $\beta^{-1}\mu \leq \top^{-1} = \varepsilon$ , whence either  $\beta^{-1} = \varepsilon$  or  $\mu = \varepsilon$ . But, if  $\beta^{-1} = \varepsilon$ , then  $\beta = \top$ , whence  $\lambda\mu \leq \beta$  and, on the other hand, if  $\mu = \varepsilon$ , then  $\lambda\mu = \lambda\varepsilon \leq \beta$ . Thus in either case (27) implies (26).

Case (III):  $\beta = \varepsilon$  or  $\beta = \top$ : the proof of the equivalence (26)  $\Leftrightarrow$  (27) reduces to the above proofs of the cases  $\lambda = \varepsilon$  respectively  $\lambda = \top$ , since (26) and (27) are symmetric (by interchanging  $\lambda$  and  $\beta$  with  $\beta^{-1}$  and  $\lambda^{-1}$  respectively).

Case (IV):  $\mu = \varepsilon$ . Then (26) means that  $\varepsilon = \lambda\varepsilon \leq \beta$ , which is true for all  $\lambda, \beta$ , and (27) means that  $\varepsilon = \beta^{-1}\mu \leq \lambda^{-1}$ , which is also true for all  $\lambda, \beta$ . Hence (26)  $\Leftrightarrow$  (27).

Case (Va):  $\mu = \top$  and  $\lambda = \varepsilon$ . Then (26) means that  $\varepsilon = \varepsilon\top \leq \beta$ , which is true for all  $\beta$ , and (27) means that  $\beta^{-1}\mu \leq \top = \lambda^{-1}$ , which is also true for all  $\beta$ . Hence (26)  $\Leftrightarrow$  (27).

Case (Vb):  $\mu = \top$  and  $\lambda \neq \varepsilon$ . Then (26) means that  $\top = \lambda\top \leq \beta$ , which implies that  $\beta = \top$ , whence  $\beta^{-1}\mu = \varepsilon\mu \leq \lambda^{-1}$ , so (26) implies (27). In the reverse direction, (27) means that  $\beta^{-1}\top = \beta^{-1}\mu \leq \lambda^{-1} < \top$ , whence  $\beta^{-1} = \varepsilon$ , so  $\beta = \top$ . Therefore  $\lambda\mu \leq \top = \beta$ , and thus (27) implies (26).

B) The inequalities (28) and (29) are equivalent if  $\lambda, \mu, \beta \in \mathcal{K} \setminus \{\varepsilon\}$ . Thus it remains to consider the cases when one of  $\lambda, \mu$  or  $\beta$  is  $\varepsilon$  or  $\top$ .

Case (I):  $\beta = \varepsilon$ . Then (28) means that  $\lambda \dot{\otimes} \mu \geq \varepsilon$ , which is true for all  $\lambda, \mu$ , and (29) means that  $\top = \varepsilon^{-1} \dot{\otimes} \mu \geq \lambda^{-1}$ , which is also true for all  $\lambda, \mu$ . Hence (28)  $\Leftrightarrow$  (29).



Case (IIa):  $\beta = \top$  and  $\mu = \top$ . Then (28) means that  $\top = \lambda \dot{\otimes} \top \geq \top$ , which is true for all  $\lambda$  and (29) means that  $\top = \varepsilon \dot{\otimes} \top \geq \lambda^{-1}$ , which is also true for all  $\lambda$ . Hence (28)  $\Leftrightarrow$  (29).

Case (IIb):  $\beta = \top$  and  $\mu \neq \top$ . Then (28) means that  $\lambda \dot{\otimes} \mu \geq \top$ , which implies that  $\lambda = \top$  (since  $\mu \neq \top$ ), whence  $\beta^{-1} \dot{\otimes} \mu \geq \varepsilon = \lambda^{-1}$ , so (28) implies (29). In the reverse direction, (29) means that  $\varepsilon = \varepsilon \mu = \varepsilon \dot{\otimes} \mu \geq \lambda^{-1}$ , whence  $\lambda^{-1} = \varepsilon$ , so  $\lambda \dot{\otimes} \mu = \top \dot{\otimes} \mu \geq \beta$ , and thus (29) implies (28).

Case (III): If  $\lambda = \varepsilon$  or  $\lambda = \top$ : the proof of the equivalence (28)  $\Leftrightarrow$  (29) reduces to the above proofs of the cases  $\beta = \varepsilon$  respectively  $\beta = \top$ , since (28) and (29) are symmetric (by interchanging  $\lambda$  and  $\beta$  with  $\beta^{-1}$  and  $\lambda^{-1}$  respectively).

Case (IVa):  $\mu = \varepsilon$  and  $\lambda = \top$ . Then (28) means that  $\top = \top \dot{\otimes} \varepsilon \geq \beta$ , which is true for all  $\beta$ , and (29) means that  $\beta^{-1} \dot{\otimes} \varepsilon \geq \lambda^{-1} = \varepsilon$ , which is also true for all  $\beta$ . Hence (28)  $\Leftrightarrow$  (29).

Case (IVb):  $\mu = \varepsilon$  and  $\lambda \neq \top$ . Then (28) means that  $\varepsilon = \lambda \dot{\otimes} \varepsilon \geq \beta$ , which implies that  $\beta = \varepsilon$ , whence  $\top = \beta^{-1} = \beta^{-1} \dot{\otimes} \mu \geq \lambda^{-1}$ , and thus (28) implies (29). In the reverse direction, (29) means that  $\varepsilon = \beta^{-1} \dot{\otimes} \varepsilon \geq \lambda^{-1}$ , whence  $\lambda^{-1} = \varepsilon$ , so  $\lambda \dot{\otimes} \mu = \top \dot{\otimes} \mu \geq \beta$ , and thus (29) implies (28).

Case (V):  $\mu = \top$ . Then (28) means that  $\top = \lambda \dot{\otimes} \top \geq \beta$ , which is true for all  $\lambda, \beta$ , and (29) means that  $\top = \beta^{-1} \dot{\otimes} \top \geq \lambda$ , which is also true for all  $\lambda, \beta$ . Hence (28)  $\Leftrightarrow$  (29).  $\square$

**Remark 12** a) In general, for  $\lambda, \mu, \beta \in \overline{\mathcal{K}}$ , the inequality (26) is not equivalent to the inequality

$$\mu \leq \lambda^{-1} \beta.$$

A counterexample is obtained by taking  $\lambda = \beta = \varepsilon$  and  $\mu \in \mathcal{K} \setminus \{\varepsilon\}$ . Indeed, then (26) becomes  $\varepsilon \leq \varepsilon$ , thus it is true, while the second inequality becomes  $\mu \leq \top \varepsilon = \varepsilon$ , which is false.

b) In general, for  $\lambda, \mu, \beta \in \overline{\mathcal{K}}$ , the inequality (28) is not equivalent to the inequality

$$\mu \geq \lambda^{-1} \dot{\otimes} \beta.$$

A counterexample is obtained by taking  $\lambda = \beta = \top$  and  $\mu \in \mathcal{K} \setminus \{\varepsilon\}$ . Indeed, then (28) becomes  $\top \geq \top$ , which is true, while the second inequality becomes  $\mu \geq \top$ , which is false.

### 3 Characterizations of topical and anti-topical functions $f : X \rightarrow \overline{\mathcal{K}}$ in terms of some inequalities

**Definition 13** Let  $(X, \mathcal{K})$  be a pair satisfying (A0'), (A1), and let  $\overline{\mathcal{K}} = \mathcal{K} \cup \{\top\}$  be the minimal enlargement of  $\mathcal{K}$ . A function  $f : X \rightarrow \overline{\mathcal{K}}$  is said to be

a) *increasing* (resp. *decreasing*), if  $x', x'' \in X, x' \leq x''$  imply  $f(x') \leq f(x'')$  (resp.  $f(x') \geq f(x'')$ );

b) *homogeneous* (resp. *anti-homogeneous*), if

$$f(\lambda x) = \lambda f(x), \quad \forall x \in X, \forall \lambda \in \mathcal{K} \quad (30)$$

(resp. if

$$f(\lambda x) = \lambda^{-1} \dot{\otimes} f(x), \quad \forall x \in X, \forall \lambda \in \mathcal{K}; \quad (31)$$

c) *topical* (resp. *anti-topical*), if it is increasing and homogeneous (resp. decreasing and anti-homogeneous).

**Lemma 14** a) Let  $f : X \rightarrow \overline{\mathcal{K}}$  be a topical function and for each  $y \in X$  let  $t_y : X \rightarrow \overline{\mathcal{K}}$  be the function defined by

$$t_y(x) = f(y)x/y, \quad \forall x \in X. \quad (32)$$

Then we have

$$t_y \leq f, \quad t_y(y) = f(y). \quad (33)$$

Conversely, if  $t = t_y : X \rightarrow \overline{\mathcal{K}}$  is a function of the form

$$t(x) = t_y(x) = \alpha x/y, \quad \forall x \in X, \quad (34)$$

where  $y \in X, \alpha \in \overline{\mathcal{K}}$ , satisfying (33), then  $\alpha = f(y)$ , so  $t_y$  is equal to (32).

b) Let  $f : X \rightarrow \overline{\mathcal{K}}$  be an anti-topical function. For any  $y \in X$  define  $q_y : X \rightarrow \overline{\mathcal{K}}$  by

$$q_y(x) := (x/y)^{-1} \dot{\otimes} f(y), \quad \forall x \in X. \quad (35)$$

Then

$$q_y \geq f, \quad q_y(y) = f(y). \quad (36)$$

Conversely, if  $q = q_y : X \rightarrow \overline{\mathcal{K}}$  is a function of the form

$$q(x) = q_y(x) = (x/y)^{-1} \dot{\otimes} \alpha, \quad \forall x \in X, \quad (37)$$

where  $y \in X, \alpha \in \overline{\mathcal{K}}$ , satisfying (36), then we have  $\alpha = f(y)$ , so  $q_y$  is equal to (35).

**Proof.** a) Observe first that for every homogeneous (and hence for every topical) function  $f : X \rightarrow \overline{\mathcal{K}}$  we have

$$f(\inf X) = \varepsilon. \quad (38)$$

Indeed, by (18), the homogeneity of  $f$  and (11) we have

$$f(\inf X) = f(\varepsilon \inf X) = \varepsilon f(\inf X) = \varepsilon.$$

Now let  $f$  be topical. If  $y \in X \setminus \{\inf X\}$ , then since  $f$  is topical, by (32), (3) and (4) we have

$$t_y(x) = f(y)x/y = f((x/y)y) \leq f(x), \quad \forall x \in X, \quad (39)$$

$$t_y(y) = f(y)(y/y) = f(y)e = f(y). \quad (40)$$

On the other hand, if  $y = \inf X$ , then by (32), (38), (21), (11) and (6),

$$t_{\inf X}(x) = f(\inf X)(x/\inf X) = \varepsilon \top = \varepsilon \leq f(x), \quad \forall x \in X, \quad (41)$$

$$t_{\inf X}(\inf X) = f(\inf X)(\inf X/\inf X) = \varepsilon \top = \varepsilon = f(\inf X). \quad (42)$$

Conversely, if  $t_y$  is a function of the form (34) satisfying (33) and  $y \in X \setminus \{\inf X\}$ , so  $y/y = e$ , then we have  $\alpha = \alpha y/y = t_y(y) = f(y)$ . On the other hand, for  $y = \inf X$ , formulae (34), (21), (33) and (38) mean that

$$t_{\inf X}(x) = \alpha x/\inf X = \alpha \top, \quad \forall x \in X, \quad (43)$$

$$t_{\inf X} \leq f, \quad t_{\inf X}(\inf X) = f(\inf X) = \varepsilon. \quad (44)$$

Hence by (43) for  $x = \inf X$  and the second part of (44), we obtain

$$\alpha \top = t_{\inf X}(\inf X) = \varepsilon,$$

which, by (10), (11) and (38), implies that  $\alpha = \varepsilon = f(\inf X)$ .

b) Observe first that *for every anti-homogeneous (and hence for every anti-topical) function  $f : X \rightarrow \overline{K}$  we have*

$$f(\inf X) = \top. \quad (45)$$

Indeed, by (18), the anti-homogeneity of  $f$ ,  $\varepsilon^{-1} = \top$  and (9) we have

$$f(\inf X) = f(\varepsilon \inf X) = \varepsilon^{-1} \dot{\otimes} f(\inf X) = \top \dot{\otimes} f(\inf X) = \top.$$

Now let  $f$  be anti-topical. If  $y \in X \setminus \{\inf X\}$ , then by (35), the anti-topicality of  $f$ , (3) and (4) we have

$$q_y(x) = (x/y)^{-1} \dot{\otimes} f(y) = f((x/y)y) \geq f(x), \quad \forall x \in X, \quad (46)$$

$$q_y(y) = (y/y)^{-1} \dot{\otimes} f(y) = e^{-1} \dot{\otimes} f(y) = f(y). \quad (47)$$

On the other hand, if  $y = \inf X$ , then by (35), (45), (9), (6) and (21),

$$\begin{aligned} q_{\inf X}(x) &= (x/\inf X)^{-1} \dot{\otimes} f(\inf X) = (x/\inf X)^{-1} \dot{\otimes} \top = \top \\ &\geq f(x), \quad \forall x \in X, \\ q_{\inf X}(\inf X) &= (\inf X/\inf X)^{-1} \dot{\otimes} f(\inf X) = \top^{-1} \dot{\otimes} \top = \top = f(\inf X) \end{aligned} \quad (48)$$

Conversely, if  $q_y$  is a function of the form (37), satisfying (36) and if  $y \in X \setminus \{\inf X\}$ , so  $y/y = e$ , then by the second part of (36) we have  $\alpha = (y/y)^{-1} \dot{\otimes} \alpha = q_y(y) = f(y)$ . On the other hand, for  $y = \inf X$ , formulae (37), (21), (36) and (45) mean that

$$q_{\inf X}(x) = (x/\inf X)^{-1} \dot{\otimes} \alpha = \top^{-1} \dot{\otimes} \alpha = \varepsilon \dot{\otimes} \alpha, \quad \forall x \in X, \quad (50)$$

$$q_{\inf X} \geq f, \quad q_{\inf X}(\inf X) = f(\inf X) = \top, \quad (51)$$

whence by (50) for  $x = \inf X$  and the second part of (51), we obtain

$$\varepsilon \dot{\otimes} \alpha = q_{\inf X}(\inf X) = \top,$$

which, by (8), (11), (9) and (45), implies that  $\alpha = \top = f(\inf X)$ .  $\square$

In [14, Theorem 5] we have shown that if  $(X, \mathcal{K})$  is a pair satisfying  $(A0')$ ,  $(A1)$ , then a function  $f : X \rightarrow \mathcal{K}$  is topical if and only if  $f(\inf X) = \varepsilon$  and

$$f(y)x/y \leq f(x), \quad \forall x \in X, \forall y \in X \setminus \{\inf X\} \quad (52)$$

(where the condition on  $y$  was needed in order to be able to define  $x/y$ ). A similar characterization of topical functions  $f : X \rightarrow \mathcal{K}$  in which the inequality  $f(y)x/y \leq f(x)$  is replaced by  $f(y)s_{y,d}(x) \leq f(x), \forall x \in X, \forall y \in X \setminus \{\inf X\}, \forall d \in \mathcal{K}$ , where

$$\begin{aligned} s_{y,d}(x) &: = \inf\{x/y, d\} = \inf\{\max\{\lambda \in \mathcal{K} \mid \lambda y \leq x\}, d\}, \\ \forall x &\in X, \forall y \in X \setminus \{\inf X\}, \forall d \in \mathcal{K}, \end{aligned} \quad (53)$$

has been given in [14, Theorem 16]. Now, with the aid of the multiplication  $\otimes$  on  $\overline{\mathcal{K}}$  defined above, we shall extend these results to functions  $f : X \rightarrow \overline{\mathcal{K}}$ , replacing the conditions  $y \in X \setminus \{\inf X\}$  and  $d \in \mathcal{K}$  by  $y \in X$  and  $d \in \overline{\mathcal{K}}$  respectively. To this end, for the case of topical functions we shall extend  $s_{y,d}(x)$  to all  $y \in X$  and  $d \in \overline{\mathcal{K}}$  by (53) and

$$s_{\inf X, d}(x) : = \inf\{x/\inf X, d\} = \inf\{\top, d\} = d, \quad \forall x \in X, \forall d \in \overline{\mathcal{K}}, \quad (54)$$

$$s_{y, \top}(x) : = \inf\{x/y, \top\} = x/y, \quad \forall x \in X, \forall y \in X. \quad (55)$$

Moreover, we shall also give corresponding characterizations of anti-topical functions, using the multiplication  $\dot{\otimes}$  defined above and the functions

$$\overline{s}_{y,d}(x) := \sup\{(x/y)^{-1}, d\}, \quad \forall x \in X, \forall y \in X, \forall d \in \overline{\mathcal{K}}; \quad (56)$$

note that in particular for the extreme values  $d = \varepsilon$  and  $d = \top$  in (56) we have, respectively,

$$\overline{s}_{y,\varepsilon}(x) = \sup\{(x/y)^{-1}, \varepsilon\} = (x/y)^{-1}, \quad \forall x \in X, \forall y \in X, \quad (57)$$

$$\overline{s}_{y,\top}(x) = \sup\{(x/y)^{-1}, \top\} = \top, \quad \forall x \in X, \forall y \in X. \quad (58)$$

**Theorem 15** *Let  $(X, \mathcal{K})$  be a pair that satisfies  $(A0')$ ,  $(A1)$ .*

a) *For a function  $f : X \rightarrow \overline{\mathcal{K}}$  the following statements are equivalent:*

1°.  *$f$  is topical.*

2°. *We have (38) and*

$$f(y)x/y \leq f(x), \quad \forall x \in X, \forall y \in X. \quad (59)$$

3°. *We have (38) and*

$$f(y)s_{y,d}(x) \leq f(x), \quad \forall x \in X, \forall y \in X, \forall d \in \overline{\mathcal{K}}. \quad (60)$$

b) For a function  $f : X \rightarrow \overline{\mathcal{K}}$  the following statements are equivalent:

1°.  $f$  is anti-topical.

2°. We have (45) and

$$f(y) \dot{\otimes} (x/y)^{-1} \geq f(x), \quad \forall x \in X, \forall y \in X. \quad (61)$$

3°. We have (45) and

$$f(y) \dot{\otimes} \overline{s}_{y,d}(x) \geq f(x), \quad \forall x \in X, \forall y \in X, \forall d \in \overline{\mathcal{K}}. \quad (62)$$

**Proof.** a) The implication  $1^\circ \Rightarrow 2^\circ$  follows from Lemma 14 a) and its proof.

$2^\circ \Rightarrow 1^\circ$ . Assume  $2^\circ$ . We need to show that  $f$  is increasing and homogeneous.

Assume that  $x, y \in X, y \leq x$ . Then  $e \leq x/y$  and due to (16) and (59) one has

$$f(y) = f(y)e \leq f(y)x/y \leq f(x),$$

so  $f$  is increasing.

Assume now that  $x \in X \setminus \{\inf X\}$  and  $\lambda \in \mathcal{K} \setminus \{\varepsilon\}$ , so  $\lambda x \in X \setminus \{\inf X\}, x/x = e$  (by (4)) and  $\lambda\lambda^{-1} = e$ . Then by (59) with  $y = \lambda x$  we have  $f(\lambda x)x/\lambda x \leq f(x)$ , whence using also (5),

$$f(\lambda x) = \lambda f(\lambda x)(\lambda^{-1}x)/x = \lambda f(\lambda x)x/\lambda x \leq \lambda f(x). \quad (63)$$

On the other hand, for  $x = \inf X, \lambda \in \mathcal{K}$ , we have, by (18), (38) and (6),

$$f(\lambda \inf X) = f(\inf X) = \varepsilon = \lambda f(\inf X). \quad (64)$$

Moreover, if  $x \in X, \lambda = \varepsilon$ , then by  $\varepsilon x = \inf X, \forall x \in X$ , and (38) we have

$$f(\varepsilon x) = f(\inf X) = \varepsilon = \varepsilon f(x). \quad (65)$$

Furthermore, if  $y \in X \setminus \{\inf X\}, \lambda \in \mathcal{K}$ , then by (4) and (59) with  $x = \lambda y$  we have

$$\lambda f(y) = f(y)(\lambda y)/y \leq f(\lambda y). \quad (66)$$

On the other hand, for  $y = \inf X, \lambda \in \mathcal{K}$ , we have

$$\lambda f(\inf X) = \lambda \varepsilon = \varepsilon = f(\lambda \inf X). \quad (67)$$

From (63)–(67) it follows that  $f$  is homogeneous.

$2^\circ \Rightarrow 3^\circ$ . Assume  $2^\circ$ . If  $x \in X, y \in X \setminus \{\inf X\}$ , then for any  $d \in \overline{\mathcal{K}}$  we have, by (53) and  $2^\circ$ ,

$$f(y)s_{y,d}(x) = f(y) \inf\{x/y, d\} \leq f(y)x/y \leq f(x).$$

If  $y = \inf X$ , then by (38) we have

$$f(\inf X)s_{\inf X,d}(x) = \varepsilon s_{\inf X,d}(x) = \varepsilon \leq f(x), \quad \forall x \in X, \forall d \in \overline{\mathcal{K}}.$$

$3^\circ \Rightarrow 2^\circ$ . Assume  $3^\circ$ . If  $x \in X, y \in X \setminus \{\inf X\}$ , then by  $3^\circ$  with any  $d \geq x/y$  we obtain

$$f(y)x/y = f(y) \inf\{x/y, d\} = f(y)s_{y,d}(x) \leq f(x).$$

Finally, if  $x \in X, y = \inf X$ , then  $f(\inf X)(x/\inf X) = \varepsilon \top = \varepsilon \leq f(x)$ .

b) The implication  $1^\circ \Rightarrow 2^\circ$  follows from Lemma 14 b) and its proof.

$2^\circ \Rightarrow 1^\circ$ . Assume  $2^\circ$ . We need to show that  $f$  is decreasing and anti-homogeneous.

Let  $x, y \in X, y \leq x$ . If  $y \in X \setminus \{\inf X\}$ , then  $e \leq x/y$ , so  $e \geq (x/y)^{-1}$ , and hence, due to (17) and (61), one has:

$$f(y) = f(y) \dot{\otimes} e \geq f(y) \dot{\otimes} (x/y)^{-1} \geq f(x).$$

On the other hand, if  $y = \inf X$ , then by (45) and (6) we have  $f(\inf X) = \top \geq f(x)$ , so  $f$  is decreasing.

Assume now that  $x \in X \setminus \{\inf X\}$  and  $\lambda \in \mathcal{K} \setminus \{\varepsilon\}$ . Then by (61) with  $y = \lambda x$  we have  $f(\lambda x) \dot{\otimes} (x/\lambda x)^{-1} \geq f(x)$ , whence, since  $(x/\lambda x)^{-1} = (\lambda^{-1}x/x)^{-1} = \lambda$  (by (5) and (4)), we obtain  $f(\lambda x) \dot{\otimes} \lambda \geq f(x)$ . Hence, since  $\lambda \in \mathcal{K} \setminus \{\varepsilon\}$ , it follows that

$$f(\lambda x) \geq \lambda^{-1} \dot{\otimes} f(x). \quad (68)$$

On the other hand, if  $x \in X \setminus \{\inf X\}$  and  $\lambda = \varepsilon$ , then by (45), (9) and (12) we have

$$f(\varepsilon x) = f(\inf X) = \top = \top \dot{\otimes} f(x) = \varepsilon^{-1} \dot{\otimes} f(x). \quad (69)$$

Furthermore, if  $x = \inf X, \lambda \in \mathcal{K}$ , then by (18), (45) and (6), we obtain

$$f(\lambda \inf X) = f(\inf X) = \top \geq \lambda^{-1} \dot{\otimes} f(\inf X). \quad (70)$$

Assume now that  $y \in X \setminus \{\inf X\}$  and  $\lambda \in \mathcal{K} \setminus \{\varepsilon\}$ . Then by (61) with  $x = \lambda y$  we have  $f(y) \dot{\otimes} (\lambda y/y)^{-1} \geq f(\lambda y)$ , whence by (4),

$$f(\lambda y) \leq f(y) \dot{\otimes} (\lambda y/y)^{-1} = \lambda^{-1} \dot{\otimes} f(y). \quad (71)$$

On the other hand, if  $y = \inf X$  and  $\lambda \in \mathcal{K} \setminus \{\varepsilon\}$ , then by (6), (9) and (45),

$$f(\lambda \inf X) \leq \top = \lambda^{-1} \dot{\otimes} \top = \lambda^{-1} \dot{\otimes} f(\inf X). \quad (72)$$

Finally, assume that  $\lambda = \varepsilon$ . Then by  $\varepsilon x = \inf X, \forall x \in X$ , (45) and (9) we have

$$f(\varepsilon x) = f(\inf X) = \top = \top \dot{\otimes} f(x) = \varepsilon^{-1} \dot{\otimes} f(x), \quad \forall x \in X. \quad (73)$$

From (68)–(73) it follows that  $f$  is anti-homogeneous.

$2^\circ \Rightarrow 3^\circ$ . Assume  $2^\circ$  and let  $y \in X, x \in X \setminus \{\inf X\}$ . Then for any  $d \in \overline{\mathcal{K}}$  we have, by (56) and  $2^\circ$ ,

$$f(y) \dot{\otimes} \overline{s}_{y,d}(x) = f(y) \dot{\otimes} \sup\{(x/y)^{-1}, d\} \geq f(y) \dot{\otimes} (x/y)^{-1} \geq f(x).$$

On the other hand, if  $y \in X, x = \inf X$ , then by (56) and (21), for any  $d \in \overline{\mathcal{K}}$  we have

$$\begin{aligned} f(y) \dot{\otimes} \overline{s}_{y,d}(\inf X) &= f(y) \dot{\otimes} \sup\{(\inf X)/y)^{-1}, d\} \\ &= \begin{cases} f(y) \dot{\otimes} \sup\{\varepsilon^{-1}, d\} = f(y) \dot{\otimes} \top = \top & \text{if } y \neq \inf X \\ f(\inf X) \dot{\otimes} \sup\{\top^{-1}, d\} = \top \dot{\otimes} d = \top & \text{if } y = \inf X, \end{cases} \end{aligned}$$

whence, by (45),  $f(y) \dot{\otimes} \overline{s}_{y,d}(\inf X) = \top = f(\inf X)$ .

$3^\circ \Rightarrow 2^\circ$ . Assume  $3^\circ$ . If  $y \in X, x \in X \setminus \{\inf X\}$ , then by  $3^\circ$  with any  $d \in \overline{\mathcal{K}}$  such that  $d \leq (x/y)^{-1}$  we obtain

$$f(y) \dot{\otimes} (x/y)^{-1} = f(y) \dot{\otimes} \sup\{(x/y)^{-1}, d\} = f(y) \dot{\otimes} \overline{s}_{y,d}(x) \geq f(x).$$

Furthermore, if  $y \in X \setminus \{\inf X\}, x = \inf X$ , then by (21) and (45),

$$f(y) \dot{\otimes} (\inf X/y)^{-1} = f(y) \dot{\otimes} \varepsilon^{-1} = f(y) \dot{\otimes} \top = \top = f(\inf X).$$

Finally, if  $y = x = \inf X$ , then by (45) and  $(\inf X/\inf X)^{-1} = \top^{-1} = \varepsilon$  we have

$$f(\inf X) \dot{\otimes} (\inf X/\inf X)^{-1} = \top \dot{\otimes} \varepsilon = \top = f(\inf X). \quad \square$$

**Remark 16** In the statements of Theorem 15 one can replace, equivalently, the inequalities by equalities. For example in Theorem 15 a) Statement  $2^\circ$  can be replaced, equivalently, by

$2'$ . We have (38) and

$$\sup_{y \in X} f(y)x/y = f(x), \quad \forall x \in X. \quad (74)$$

Indeed, for each  $x \in X \setminus \{\inf X\}$  the sup in (74) is attained at  $y = x$ . On the other hand, for  $x = \inf X$  we have  $\sup_{y \in X} f(y)(\inf X/y) = \varepsilon = f(\inf X)$  (by (38)). Thus  $2^\circ \Rightarrow 2'$ . The reverse implication is obvious.

Similarly, in Theorem 15 a) Statement  $3^\circ$  can be replaced, equivalently, by:

$3'$ . We have (38) and

$$\sup_{(y,d) \in X \times \overline{\mathcal{K}}} f(y)s_{y,d}(x) = f(x), \quad \forall x \in X. \quad (75)$$

Indeed, for each  $x \in X \setminus \{\inf X\}$  the sup in (75) is attained at  $y = x, d \geq e$  (since then  $f(x)s_{x,d}(x) = f(x)\inf\{x/x, d\} = f(x)e = f(x)$ ); furthermore, for  $x = \inf X, y \in X \setminus \{\inf X\}$ , so  $\inf X/y = \varepsilon$ , and any  $d \in \overline{\mathcal{K}}$ , we have, by (22) and (38),

$$\sup_{y \in X} f(y)s_{y,d}(\inf X) = \sup_{y \in X} f(y)\inf\{\inf X/y, d\} = \sup_{y \in X} f(y)\varepsilon = \varepsilon = f(\inf X);$$

finally, if  $x = y = \inf X$ , then for any  $d \in \overline{\mathcal{K}}$  we have, by (38),

$$f(\inf X)s_{\inf X,d}(\inf X) = \varepsilon s_{\inf X,d}(\inf X) = \varepsilon = f(\inf X).$$

Thus  $3^\circ \Rightarrow 3'$ . The reverse implication is obvious. The cases of Theorem 15b) are similar.

**Corollary 17** a) If  $f$  is topical and for some  $y \in X$  we have  $f(y) = \top$ , then for each  $x \in X$  either  $f(x) = \top$  or  $x/y = \varepsilon$ .

b) If  $f$  is anti-topical and for some  $y \in X$  we have  $f(y) = \varepsilon$ , then for each  $x \in X$  either  $f(x) = \varepsilon$  or  $(x/y)^{-1} = \top$ .

c) A function  $f : X \rightarrow \overline{\mathcal{K}}$  cannot be simultaneously topical and anti-topical.

**Proof.** a) If  $f$  is topical,  $x, y \in X$  and  $f(y) = \top$ , then by Theorem 15a), implication  $1^\circ \Rightarrow 2^\circ$ , we have  $\top x/y \leq f(x)$ . Hence if  $f(x) \neq \top$ , then by (10) we obtain  $x/y = \varepsilon$ .

The proof of part b) is similar, mutatis mutandis.

c) This follows from the fact that values of  $f$  at  $\inf X$  are  $f(\inf X) = \varepsilon$  and  $f(\inf X) = \top$  for topical and anti-topical functions respectively, and  $\top \neq \varepsilon$ .  $\square$

**Corollary 18** Let  $(X, \mathcal{K})$  be a pair that satisfies  $(A0')$ ,  $(A1)$ .

a) For a topical function  $f : X \rightarrow \overline{\mathcal{K}}$  one has

$$f(x)^{-1}x/y \leq f(y)^{-1}, \quad \forall x \in X, y \in X. \quad (76)$$

b) For an anti-topical function  $f : X \rightarrow \overline{\mathcal{K}}$  one has

$$f(x)^{-1} \dot{\otimes} (x/y)^{-1} \geq f(y)^{-1}, \quad \forall x \in X, \forall y \in X. \quad (77)$$

**Proof.** Part a) follows from Theorem 15 a) and Lemma 11 a).

Part b) follows from Theorem 15 b) and Lemma 11 b).  $\square$

The following Corollary of Theorem 15 gives characterizations of the functions  $f$  that satisfy the inequalities (59)-(62):

**Corollary 19** Let  $(X, \mathcal{K})$  be a pair that satisfies  $(A0')$ ,  $(A1)$ .

a) For a function  $f : X \rightarrow \overline{\mathcal{K}}$  the following statements are equivalent:

1°. We have (59).

2°. We have (60).

3°. Either  $f$  is topical or  $f \equiv \top$ .

b) For a function  $f : X \rightarrow \overline{\mathcal{K}}$  the following statements are equivalent:

1°. We have (61).

2°. We have (62).

3°. Either  $f$  is anti-topical or  $f \equiv \varepsilon$ .

**Proof.** a)  $1^\circ \Rightarrow 3^\circ$ . Assume that we have  $1^\circ$  and  $f$  is not topical, so  $f(\inf X) \neq \varepsilon$  (by Theorem 15a)). Then by (21) and (59) (applied to  $y = \inf X$ ) we have

$$f(\inf X)\top = f(\inf X)x/\inf X \leq f(x), \quad \forall x \in X,$$

whence, since  $f(\inf X)\top = \top$  (by  $f(\inf X) \neq \varepsilon$ ), it follows that  $\top \leq f(x), \forall x \in X$ , and hence  $f \equiv \top$ .



$3^\circ \Rightarrow 1^\circ$ . Assume  $3^\circ$ . If  $f$  is topical, then we have (59) by Theorem 15a). On the other hand, if  $f \equiv \top$ , then (59) holds since  $\top$  is the greatest element of  $\overline{\mathcal{K}}$ .

$2^\circ \Rightarrow 3^\circ$ . If  $2^\circ$  holds, then by (55) and (60) applied to  $d = \top$  we have

$$f(y)x/y = f(y)s_{y,\top}(x) \leq f(x), \quad \forall x \in X, \forall y \in X,$$

so we have (59). Hence by the implication  $1^\circ \Rightarrow 3^\circ$  proved above, we obtain  $3^\circ$ .

Finally, the proof of the implication  $3^\circ \Rightarrow 2^\circ$  is obtained replacing in the above proof of the implication  $3^\circ \Rightarrow 1^\circ$ , (59) by (60).

b)  $1^\circ \Rightarrow 3^\circ$ . Assume that we have  $1^\circ$  and  $f$  is not anti-topical, so  $f(\inf X) \neq \top$  (by Theorem 15b)). Then by (21),  $\top^{-1} = \varepsilon$  and (61) applied to  $y = \inf X$ , we have

$$f(\inf X) \dot{\otimes} \varepsilon = f(\inf X) \dot{\otimes} (x/\inf X)^{-1} \geq f(x), \quad \forall x \in X,$$

whence, since  $f(\inf X) \dot{\otimes} \varepsilon = \varepsilon$  (by  $f(\inf X) \neq \top$ ), it follows that  $\varepsilon \geq f(x), \forall x \in X$ , and hence  $f \equiv \varepsilon$ .

$3^\circ \Rightarrow 1^\circ$ . Assume  $3^\circ$ . If  $f$  is anti-topical, then we have (61) by Theorem 15b). On the other hand, if  $f \equiv \varepsilon$ , then (61) holds since  $\varepsilon$  is the smallest element of  $\overline{\mathcal{K}}$ .

Similarly, the proofs of the implications  $2^\circ \Rightarrow 3^\circ$  and  $3^\circ \Rightarrow 2^\circ$  are dual to the corresponding implications of the above proofs of part a).  $\square$

**Remark 20** The constant function  $f \equiv \top$  is not homogeneous, and hence not topical. Indeed, we have  $\top(\varepsilon x) = \top$ , but  $\varepsilon \top(x) = \varepsilon, \forall x \in X$ . However,  $f \equiv \top$  is anti-topical, since  $\top(\lambda x) = \top, \lambda^{-1} \dot{\otimes} \top(x) = \top, \forall x \in X, \forall \lambda \in \mathcal{K}$ . Let us also mention that the constant function  $f \equiv \varepsilon$  is topical and hence not anti-topical.

## 4 Characterizations of topical and anti-topical functions $f : X \rightarrow \overline{\mathcal{K}}$ using conjugates of Fenchel-Moreau type

We recall that for two sets  $X$  and  $Y$  and a “finite coupling function”  $\pi : X \times Y \rightarrow R$ , respectively  $\pi : X \times Y \rightarrow A$ , where  $A = (A, \oplus, \otimes)$  is a conditionally complete lattice ordered group, the Fenchel-Moreau conjugate function (with respect to  $\pi$ ) of a function  $f : X \rightarrow \overline{R}$ , respectively  $f : X \rightarrow \overline{A}$ , where  $\overline{A}$  is the canonical enlargement of  $A$ , has been studied in [9, 12].

For the next characterizations of topical and anti-topical functions  $f : X \rightarrow \overline{\mathcal{K}}$  it will be convenient to introduce the following similar notion of Fenchel-Moreau conjugations:

**Definition 21** Let  $(X, \mathcal{K}), (Y, \mathcal{K})$  be two pairs satisfying  $(A0'), (A1)$ , let  $\overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$  be the minimal enlargement of  $\mathcal{K}$ , and let  $\pi : X \times Y \rightarrow \overline{\mathcal{K}}$  be a function, called “coupling function”. The *Fenchel-Moreau conjugate (with respect to  $\pi$ )* of a function  $f : X \rightarrow \overline{\mathcal{K}}$  is the function  $f^{c(\pi)} : Y \rightarrow \overline{\mathcal{K}}$  defined by

$$f^{c(\pi)}(y) := \sup_{x \in X} f(x)^{-1} \pi(x, y), \quad \forall y \in Y. \quad (78)$$

If  $\alpha \in \overline{\mathcal{K}}$  and  $X$  is any set, we shall use the same notation  $\alpha$  also for the constant function  $f(x) \equiv \alpha, \forall x \in X$ , and we shall write briefly  $f \equiv \alpha$  or  $f = \alpha$ .

**Remark 22** a) For the constant function  $f \equiv \top$  we have

$$\top^{c(\pi)}(y) = \varepsilon, \quad \forall y \in Y. \quad (79)$$

Indeed, by (12) and (11) we have

$$\top^{c(\pi)}(y) = \sup_{x \in X} (\top(x)^{-1}) \pi(x, y) = \sup_{x \in X} \varepsilon \pi(x, y) = \varepsilon, \quad \forall y \in Y.$$

b) For the constant function  $f \equiv \varepsilon$  we have

$$\begin{aligned} \varepsilon^{c(\pi)}(y) &= \sup_{x \in X} \varepsilon(x)^{-1} \pi(x, y) = \sup_{x \in X} \top \pi(x, y) \\ &= \begin{cases} \top & \text{if } \exists x_0 \in X, \pi(x_0, y) \neq \varepsilon \\ \varepsilon & \text{if } \pi(x, y) = \varepsilon, \forall x \in X. \end{cases} \end{aligned} \quad (80)$$

c) If  $f : X \rightarrow \overline{\mathcal{K}}$  and  $x_0 \in X$  are such that  $f(x_0) = \varepsilon$  (e.g., if  $f$  is homogeneous, or in particular, topical, and  $x_0 = \inf X$ ) and if  $y_0 \in Y, \pi(x_0, y_0) \neq \varepsilon$ , then

$$f^{c(\pi)}(y_0) = \top. \quad (81)$$

Indeed, by (78) and  $\varepsilon^{-1} = \top$  we have  $f^{c(\pi)}(y_0) \geq f(x_0)^{-1} \pi(x_0, y_0) = \top \pi(x_0, y_0)$ , whence by  $\pi(x_0, y_0) \neq \varepsilon$ , (11) and (6) we obtain (81).

Here we shall be interested first in the case where  $(X, \mathcal{K})$  is a pair satisfying  $(A0')$ ,  $(A1)$  and  $Y := X$ , with the coupling function  $\pi = \varphi : X \times X \rightarrow \overline{\mathcal{K}}$  defined by

$$\varphi(x, y) := x/y = y^\diamond(x), \quad \forall x \in X, \forall y \in X. \quad (82)$$

In a different context, related to the study of increasing positively homogeneous functions  $f : C \rightarrow R_+$  defined on a cone  $C$  of a locally convex space  $X$  endowed with the order induced by the closure  $\overline{C}$  of  $C$ , with values in  $R_+ = R \cup \{+\infty\}$ , the coupling function (82) has been considered in [6].

Second, we shall be interested in the coupling function  $\pi = \psi : X \times (X \times \overline{\mathcal{K}}) \rightarrow \overline{\mathcal{K}}$  defined by

$$\psi(x, (y, d)) := \inf\{x/y, d\} = s_{y,d}(x), \quad \forall x \in X, \forall y \in X, \forall d \in \overline{\mathcal{K}}. \quad (83)$$

For  $\pi = \varphi$  and  $\pi = \psi$  Definition 21 leads to:

**Definition 23** If  $(X, \mathcal{K})$  is a pair satisfying  $(A0'), (A1)$ , the Fenchel-Moreau conjugate of a function  $f : X \rightarrow \overline{\mathcal{K}}$

a) associated to the coupling function  $\varphi$  of (82), or briefly, the  $\varphi$ -conjugate of  $f$ , is the function  $f^{c(\varphi)} : X \rightarrow \overline{\mathcal{K}}$  defined by

$$f^{c(\varphi)}(y) := \sup_{x \in X} f(x)^{-1} x/y, \quad \forall y \in X; \quad (84)$$

b) associated to the coupling function  $\psi$  of (83), or briefly, the  $\psi$ -conjugate of  $f$ , is the function  $f^{c(\psi)} : X \times \overline{\mathcal{K}} \rightarrow \overline{\mathcal{K}}$  defined by

$$f^{c(\psi)}(y, d) = \sup_{x \in X} f(x)^{-1} s_{y,d}(x), \quad \forall y \in X, \forall d \in \overline{\mathcal{K}}. \quad (85)$$

**Remark 24** a) For any function  $f : X \rightarrow \overline{\mathcal{K}}$ , the conjugate function  $f^{c(\varphi)} : X \rightarrow \overline{\mathcal{K}}$  is decreasing (i.e.  $y_1 \leq y_2 \Rightarrow f^{c(\varphi)}(y_1) \geq f^{c(\varphi)}(y_2)$ ) and anti-homogeneous (i.e.  $f^{c(\varphi)}(\lambda y) = \lambda^{-1} \dot{\otimes} f^{c(\varphi)}(y)$  for all  $y \in X$  and  $\lambda \in \mathcal{K}$ , since by Lemma 8 we have

$$\begin{aligned} f^{c(\varphi)}(\lambda y) &= \sup_{x \in X} f(x)^{-1} x / \lambda y = \sup_{x \in X} f(x)^{-1} \lambda^{-1} \dot{\otimes} x / y \\ &= \lambda^{-1} \dot{\otimes} \sup_{x \in X} f(x)^{-1} x / y = \lambda^{-1} \dot{\otimes} f^{c(\varphi)}(y), \end{aligned}$$

so  $f^{c(\varphi)}$  is anti-topical. This should be compared with the situation for the conjugates of  $f$  with respect to the so-called “additive min-type coupling functions”  $\pi_\mu : R_{\max}^n \times R_{\max}^n \rightarrow R_{\max}$  defined [9, 11] by

$$\pi_\mu(x, y) = \min_{1 \leq i \leq n} (x_i + y_i), \quad \forall x = (x_i) \in R_{\max}^n, \forall y = (y_i) \in R_{\max}^n, \quad (86)$$

with  $+$  denoting the usual addition on  $R_{\max}$ , and respectively  $\pi_\mu : A^n \times A^n \rightarrow A$ , where  $A$  is a conditionally complete lattice ordered group, defined [12] by

$$\pi_\mu(x, y) := \inf_{1 \leq i \leq n} (x_i \otimes y_i), \quad \forall x = (x_i) \in A^n, \forall y = (y_i) \in A^n. \quad (87)$$

For example, in the latter case the conjugate  $f^{c(\pi_\mu)} : A^n \rightarrow \overline{A}$  of a function  $f : A^n \rightarrow \overline{A}$  (where  $\overline{A}$  is the canonical enlargement of  $A$ ) is a  $\otimes$ -topical function (see [12, Proposition 6.1]). Note that for  $X = A^n$  and  $\varphi, \pi_\mu$  of (82), (87) we have

$$\varphi(x, y) = x / y = \inf_{1 \leq i \leq n} (x_i \otimes (y_i^{-1})) = \pi_\mu(x, y^{-1}), \quad \forall x \in A^n, \forall y \in A^n, \quad (88)$$

where  $y^{-1} := (y_i^{-1})_{1 \leq i \leq n}$  (see e.g. [1, Remark 2.4(a)] for  $R_{\max}^n$ ).

b) By (53) (extended to all  $y \in X$ ) and (55), for any  $f : X \rightarrow \overline{\mathcal{K}}$  and the extreme values  $d = \varepsilon$  and  $d = \top$  in (85) we have, respectively,

$$f^{c(\psi)}(y, \varepsilon) = \sup_{x \in X} f(x)^{-1} s_{y,\varepsilon}(x) = \sup_{x \in X} f(x)^{-1} \varepsilon = \varepsilon, \quad \forall y \in X, \quad (89)$$

$$f^{c(\psi)}(y, \top) = \sup_{x \in X} f(x)^{-1} s_{y,\top}(x) = \sup_{x \in X} f(x)^{-1} x / y = f^{c(\varphi)}(y), \quad \forall y \in X. \quad (90)$$

**Theorem 25** Let  $(X, \mathcal{K})$  be a pair satisfying assumptions  $(A0')$  and  $(A1)$ . For a function  $f : X \rightarrow \overline{\mathcal{K}}$  the following statements are equivalent:

1°.  $f$  is topical.

2°. We have (38) and

$$f^{c(\varphi)}(y) = f(y)^{-1}, \quad \forall y \in X. \quad (91)$$

3°. We have (38) and

$$f^{c(\psi)}(y, d) = f(y)^{-1}, \quad \forall y \in X, \forall d \in \overline{\mathcal{K}} \setminus \{\varepsilon\}. \quad (92)$$

**Proof.**  $1^\circ \Rightarrow 2^\circ$ . If  $1^\circ$  holds, then by Theorem 15 a) and Corollary 18 a) we have (38) and (76). Consequently by (84) and (76) we obtain  $f^{c(\varphi)}(y) \leq f(y)^{-1}$ .

In the reverse direction, by (84) we have

$$f^{c(\varphi)}(y) = \sup_{x \in X} f(x)^{-1}x/y \geq f(y)^{-1}y/y, \quad y \in Y.$$

If  $y \neq \inf X$ , then  $y/y = e$  and  $f(y)^{-1}y/y = f(y)^{-1}$ , so we obtain (91).

If  $y = \inf X$ , then by (84), (38) and (23) we have

$$f^{c(\varphi)}(\inf X) = \sup_{x \in X} f(x)^{-1}x/\inf X \geq f(\inf X)^{-1}\inf X/\inf X = \top\top = \top,$$

whence  $f^{c(\varphi)}(\inf X) = \top = \varepsilon^{-1} = f(\inf X)^{-1}$ .

$2^\circ \Rightarrow 1^\circ$ . If  $2^\circ$  holds, then for all  $x, y \in X$  we have, by (84) and (91),  $f(x)^{-1}x/y \leq f^{c(\varphi)}(y) = f(y)^{-1}$ , whence  $f(y)x/y \leq f(x)$  (by Lemma 11). Hence, by Theorem 15 a),  $f$  is topical.

$1^\circ \Rightarrow 3^\circ$ . If  $1^\circ$  holds, then by Theorem 15 a) we have (38). Assume now  $1^\circ$  and let  $x \in X, y \in X \setminus \{\inf X\}, d \in \mathcal{K} \setminus \{\varepsilon\}$ . Then by (53) (extended to all  $y \in X$ ),  $1^\circ$  and Theorem 15 a) we have

$$f(y)s_{y,d}(x) = f(y)\inf\{x/y, d\} \leq f(y)x/y \leq f(x), \quad (93)$$

whence  $f(x)^{-1}s_{y,d}(x) \leq f(y)^{-1}$  (by Lemma 11). Hence by (85) and since  $x \in X$  has been arbitrary, we get

$$f^{c(\psi)}(y, d) = \sup_{x \in X} \{f(x)^{-1}s_{y,d}(x)\} \leq f(y)^{-1}. \quad (94)$$

On the other hand, by (53),  $y \in X \setminus \{\inf X\}$  and (4) we have

$$s_{y,d}(dy) = \inf\{dy/y, d\} = d. \quad (95)$$

Hence, by (85) and (95), for any  $f : X \rightarrow \overline{\mathcal{K}}$  (not necessarily topical) and any  $y \in X \setminus \{\inf X\}, d \in \mathcal{K} \setminus \{\varepsilon\}$  we have

$$f^{c(\psi)}(y, d) \geq f(dy)^{-1}s_{y,d}(dy) = f(dy)^{-1}d,$$

whence, since by  $1^\circ$  and  $dd^{-1} = e$  there holds

$$f(dy)^{-1}d = (df(y))^{-1}d = dd^{-1}f(y)^{-1} = f(y)^{-1},$$

we obtain the opposite inequality to (94), and hence (92) for  $y \in X \setminus \{\inf X\}, d \in \mathcal{K} \setminus \{\varepsilon\}$ .

Assume now that  $y = \inf X$  and  $d \in \mathcal{K} \setminus \{\varepsilon\}$ . Then by (21), (38) and (10),

$$\begin{aligned} f^{c(\psi)}(\inf X, d) &= \sup_{x \in X} f(x)^{-1} \inf\{x/\inf X, d\} \\ &= \sup_{x \in X} f(x)^{-1} \inf\{\top, d\} = \sup_{x \in X} f(x)^{-1} d \\ &\geq f(\inf X)^{-1} d = \varepsilon^{-1} d = \top d = \top, \end{aligned}$$

so we have  $f^{c(\psi)}(\inf X, d) = \top = \varepsilon^{-1} = f(\inf X)^{-1}$ , and hence (92) for all  $d \in \mathcal{K} \setminus \{\varepsilon\}$ .

Finally, for  $d = \top$  we have, by (90),  $f^{c(\psi)}(y, \top) = f^{c(\varphi)}(y), \forall y \in X$ , so the equalities of  $3^\circ$  for  $d = \top$  and  $2^\circ$  coincide. Hence by  $1^\circ$  and the implication  $1^\circ \Rightarrow 2^\circ$  proved above, we have (92) also for  $d = \top$ .

$3^\circ \Rightarrow 1^\circ$ . If  $3^\circ$  holds, then for all  $x, y \in X, d \in \overline{\mathcal{K}} \setminus \{\varepsilon\}$ , we have, by (85) and (92),  $f(x)^{-1} s_{y,d}(x) \leq f^{c(\psi)}(y) = f(y)^{-1}$ , whence  $f(y) s_{y,d}(x) \leq f(x)$  (by Lemma 11). Furthermore, for  $d = \varepsilon$  we have  $f(x)^{-1} s_{y,\varepsilon}(x) = \varepsilon \leq f(y)^{-1}$ , whence  $f(y) s_{y,\varepsilon}(x) \leq f(x), \forall x \in X, \forall y \in X$  (by Lemma 11). Consequently, by Theorem 15a),  $f$  is topical.  $\square$

**Remark 26** a) One cannot add  $d = \varepsilon$  to statement  $3^\circ$ , since by (89) we have  $f^{c(\psi)}(y, \varepsilon) = \varepsilon, \forall y \in X$ , which shows that for a topical function  $f : X \rightarrow \overline{\mathcal{K}}$  (hence  $f \not\equiv \top$  by (38)) one cannot have  $f^{c(\psi)}(y, \varepsilon) = f(y)^{-1}, \forall y \in X$ .

b) Alternatively, one can also prove the implication  $1^\circ \Rightarrow 2^\circ$  of Theorem 25 as follows: If  $1^\circ$  holds, then by Theorem 15 a) we have (38) and (59), whence by Lemma 11 we obtain

$$\sup_{x \in X} f(x)^{-1} x/y \leq f(y)^{-1}, \quad \forall y \in Y. \quad (96)$$

Now, if  $y \neq \inf X$ , then the sup in (96) is attained at  $x = y$ , since  $f(y)^{-1} y/y = f(y)^{-1}$ .

Furthermore, if  $y = \inf X$  and there exists  $x_0 \in X$  such that  $f(x_0) \neq \top$ , or equivalently,  $f(x_0)^{-1} \neq \varepsilon$ , then

$$\sup_{x \in X} f(x)^{-1} (x/\inf X) \geq f(x_0)^{-1} (x_0/\inf X) = \top,$$

whence by (38) we obtain  $\sup_{x \in X} f(x)^{-1} (x/\inf X) = \top = \varepsilon^{-1} = f(\inf X)^{-1}$ .

Finally, if  $f \equiv \top$ , then

$$\sup_{x \in X} \top(x)^{-1} x/y = \varepsilon = \top(y)^{-1}, \quad \forall y \in Y.$$

Thus in all cases we have equality in (96).

Concerning a similar proof for the implication  $1^\circ \Rightarrow 3^\circ$  of Theorem 25 we only observe here that, as above,  $1^\circ$  implies the inequalities (93),  $\forall x \in X, \forall y \in X, \forall d \in \mathcal{K} \setminus \{\varepsilon\}$ , whence (94),  $\forall y \in X, \forall d \in \mathcal{K} \setminus \{\varepsilon\}$ . Hence, for  $y = x \in X \setminus \{\inf X\}$  one obtains  $f(x)^{-1} s_{x,d}(x) = f(x)^{-1} \inf\{x/x, d\} = f(x)^{-1}$  whenever  $e \leq d$ .

c) Theorem 25 above should be compared with [9, Theorem 5.3] and [12, Theorem 6.2]; according to the latter a function  $f : A^n \rightarrow \overline{A}$ , where  $A$  and  $\overline{A}$  are as in Remark 24a) above, is “ $\otimes$ -topical” (i.e., increasing and “ $\otimes$ -homogeneous”) if and only if for the coupling function  $\pi_\mu$  of (87) we have

$$f^{c(\pi_\mu)}(y) = [f(y^{-1})]^{-1}, \quad \forall y \in A^n. \quad (97)$$

The reason for this discrepancy between (97) and Theorem 25 is shown by formula (88) above.

d) From Corollary 19a) one obtains characterizations of the functions  $f$  that satisfy the equalities (91), (92), namely, *for a function  $f : X \rightarrow \overline{\mathcal{K}}$  the following statements are equivalent:*

- 1°. We have (91).
- 2°. We have (92).
- 3°. Either  $f$  is topical or  $f \equiv \top$ .

Indeed, if we have (91), that is,  $\sup_{x \in X} f(x)^{-1}x/y = f(y)^{-1}, \forall x \in X, \forall y \in X$ , then by Lemma 11 we obtain the inequalities (59), whence by Corollary 19a), either  $f$  is topical or  $f \equiv \top$ . In the reverse direction, if  $f$  is topical, then (91) holds by Theorem 25, while if  $f \equiv \top$ , then (91) holds since

$$\sup_{x \in X} \top(x)^{-1}x/y = \sup_{x \in X} \varepsilon x/y = \varepsilon = \top(y)^{-1}, \quad \forall y \in X.$$

On the other hand, if (92) holds, then for  $d = \top$  we have, by (90)

$$f^{c(\varphi)}(y) = f^{c(\psi)}(y, \top) = f(y)^{-1}, \quad \forall y \in X,$$

whence by the implication  $1^\circ \Rightarrow 3^\circ$  proved above, we obtain  $3^\circ$ . Conversely, if  $3^\circ$  holds and  $f$  is topical, then (92) holds by Theorem 25, while if  $f \equiv \top$ , then (92) holds since

$$\begin{aligned} \top^{c(\psi)}(y, d) &= \sup_{x \in X} \top(x)^{-1} \inf\{x/y, d\} = \sup \varepsilon \inf\{x/y, d\} = \varepsilon \\ &= \top(y)^{-1}, \quad \forall y \in X, \forall d \in \overline{\mathcal{K}} \setminus \{\varepsilon\}. \end{aligned}$$

**Lemma 27** *Let  $(X, \mathcal{K})$  be a pair satisfying assumptions (A0') and (A1) and let  $\varphi : X \times X \rightarrow \overline{\mathcal{K}}$  be the coupling function (82). Then*

- a) *For each  $y \in X$  the partial function  $\varphi(., y)$  is topical.*
- b) *For each  $x \in X$  the partial function  $\varphi(x, .)$  is anti-topical.*

**Proof.** a) Let  $y \in X$ . Then by the properties of extended residuation, for each  $x', x'', x, y \in X$  and  $\lambda \in \mathcal{K}$  we have

$$\begin{aligned} x' &\leq x'' \Rightarrow \varphi(x', y) = x'/y \leq x''/y = \varphi(x'', y), \\ x &\in X, \lambda \in \mathcal{K} \Rightarrow \varphi(\lambda x, y) = (\lambda x)/y = \lambda(x/y) = \lambda \varphi(x, y). \end{aligned}$$

b) Let  $x \in X$ . Then by the properties of extended residuation, for each  $y', y'', y, x \in X$  and  $\lambda \in \mathcal{K}$  we have

$$\begin{aligned} y' &\leq y'' \Rightarrow \varphi(x, y') = x/y' \geq x/y'' = \varphi(x, y''), \\ y &\in X, \lambda \in \mathcal{K} \Rightarrow \varphi(x, \lambda y) = x/\lambda y = \lambda^{-1} \dot{\otimes} x/y = \lambda^{-1} \dot{\otimes} \varphi(x, y). \quad \square \end{aligned}$$

**Remark 28** a) We know already part a), since in other words it says that the function  $y^\diamond : X \rightarrow \overline{\mathcal{K}}$  defined by (82) is topical (by [13] when  $y \in X \setminus \{\inf X\}$  and since  $(\inf X)^\diamond = ./\inf X \equiv \top$  by (21)).

b) Lemma 27 should be compared with the fact that for the coupling function  $\pi_\mu : A^n \times A^n \rightarrow A$  of (87) the partial functions  $\pi_\mu(., y)$  and  $\pi_\mu(x, .)$  are topical.

**Definition 29** If  $(X, \mathcal{K})$  is a pair satisfying  $(A0'), (A1)$ , the *Fenchel-Moreau lower conjugate* of a function  $f : X \rightarrow \overline{\mathcal{K}}$

a) *associated to the coupling function  $\varphi$  of (82)*, or briefly, the  $\varphi$ -*lower conjugate of  $f$* , is the function  $f^{\theta(\varphi)} : X \rightarrow \overline{\mathcal{K}}$  defined by

$$f^{\theta(\varphi)}(y) := \inf_{x \in X} \{f(x)^{-1} \dot{\otimes} (x/y)^{-1}\}, \quad \forall y \in X; \quad (98)$$

b) *associated to the coupling function  $\psi$  of (83)*, or briefly, the  $\psi$ -*lower conjugate of  $f$* , is the function  $f^{\theta(\psi)} : X \rightarrow \overline{\mathcal{K}}$  defined by

$$f^{\theta(\psi)}(y, d) := \inf_{x \in X} \{f(x)^{-1} \dot{\otimes} \overline{s}_{y,d}(x)\}, \quad \forall y \in X, \forall d \in \overline{\mathcal{K}}, \quad (99)$$

with  $\overline{s}_{y,d}(x)$  of (56).

**Remark 30** a) For any function  $f : X \rightarrow \overline{\mathcal{K}}$  such that  $f \not\equiv \varepsilon$  the lower conjugate function  $f^{\theta(\varphi)} : X \rightarrow \overline{\mathcal{K}}$  is topical. Indeed, since  $y_1 \leq y_2$  implies that  $(x/y_1)^{-1} \leq (x/y_2)^{-1}$ ,  $f^{\theta(\varphi)}$  of (98) is increasing. Furthermore, if  $y \in X \setminus \{\inf X\}$ ,  $\lambda \in \mathcal{K}$ , then by (5) we have

$$\begin{aligned} f^{\theta(\varphi)}(\lambda y) &= \inf_{x \in X} f(x)^{-1} \dot{\otimes} (x/\lambda y)^{-1} = \inf_{x \in X} f(x)^{-1} \dot{\otimes} (\lambda^{-1} x/y)^{-1} \\ &= \lambda \inf_{x \in X} f(x)^{-1} \dot{\otimes} (x/y)^{-1} = \lambda f^{\theta(\varphi)}(y); \end{aligned}$$

on the other hand, if  $y = \inf X$  then since by  $f \not\equiv \varepsilon$  there exists  $x_0 \in X$  such that  $f(x_0)^{-1} \neq \top$ , we have

$$\begin{aligned} f^{\theta(\varphi)}(\lambda \inf X) &= f^{\theta(\varphi)}(\inf X) = \inf_{x \in X} f(x)^{-1} \dot{\otimes} (x/\inf X)^{-1} \\ &\leq f(x_0)^{-1} \dot{\otimes} (x_0/\inf X)^{-1} = f(x_0)^{-1} \dot{\otimes} \varepsilon = \varepsilon, \end{aligned}$$

whence  $f^{\theta(\varphi)}(\lambda \inf X) = \varepsilon = \lambda f^{\theta(\varphi)}(\inf X)$ , so  $f^{\theta(\varphi)}$  of (98) is homogeneous, and hence topical.

However, note that for the constant function  $\varepsilon$ , that is,  $\varepsilon(x) \equiv \varepsilon, \forall x \in X$ , we have

$$\varepsilon^{\theta(\varphi)}(y) = \varepsilon^{\theta(\psi)}(y, d) = \top, \quad \forall y \in X, \forall d \in \overline{\mathcal{K}}; \quad (100)$$

indeed,

$$\varepsilon^{\theta(\varphi)}(y) = \inf_{x \in X} \{\varepsilon^{-1} \dot{\otimes} (x/y)^{-1}\} = \inf_{x \in X} \{\top \dot{\otimes} (x/y)^{-1}\} = \top, \quad \forall y \in X,$$

and the last equality of (100) follows similarly. Consequently, by Remark 20, the lower conjugate function  $\varepsilon^{\theta(\varphi)}$  is anti-topical.

b) By (56), (57) and (58), for any  $f : X \rightarrow \overline{\mathcal{K}}$  and the extreme values  $d = \varepsilon$  and  $d = \top$  in (99) we have, respectively,

$$\begin{aligned} f^{\theta(\psi)}(y, \varepsilon) &= \inf_{x \in X} f(x)^{-1} \dot{\otimes} \overline{s}_{y, \varepsilon}(x) = \inf_{x \in X} \{f(x)^{-1} \dot{\otimes} (x/y)^{-1}\} \\ &= f^{\theta(\varphi)}(y), \quad \forall y \in X, \end{aligned} \quad (101)$$

$$\begin{aligned} f^{\theta(\psi)}(y, \top) &= \inf_{x \in X} f(x)^{-1} \dot{\otimes} \overline{s}_{y, \top}(x) = \inf_{x \in X} f(x)^{-1} \dot{\otimes} \top \\ &= \top, \quad \forall y \in X. \end{aligned} \quad (102)$$

c) We have

$$f^{\theta(\varphi)}(y) = \inf_{x \in X \setminus \{\inf X\}} f(x)^{-1} \dot{\otimes} y/x, \quad \forall y \in X \setminus \{\inf X\}, \quad (103)$$

$$f^{\theta(\psi)}(y, d) = \inf_{x \in X \setminus \{\inf X\}} f(x)^{-1} \dot{\otimes} \sup\{y/x, d\}, \quad \forall y \in X \setminus \{\inf X\}, \forall d \in \overline{\mathcal{K}}. \quad (104)$$

Indeed, the equalities (103) and (104) are due to the fact that if  $y \neq \inf X$ , then by Lemma 9, (21) and (9) we have  $(\inf X/y)^{-1} = y/\inf X$  and

$$f(\inf X)^{-1} \dot{\otimes} (y/\inf X) = f(\inf X)^{-1} \dot{\otimes} \top = \top, \quad \forall y \in X, \forall d \in \overline{\mathcal{K}},$$

respectively

$$f(\inf X)^{-1} \dot{\otimes} \sup\{(y/\inf X), d\} = f(\inf X)^{-1} \dot{\otimes} \sup\{\top, d\} = \top, \quad \forall y \in X, \forall d \in \overline{\mathcal{K}}.$$

**Theorem 31** *Let  $(X, \mathcal{K})$  be a pair satisfying assumptions  $(A0')$  and  $(A1)$ . For a function  $f : X \rightarrow \overline{\mathcal{K}}$  the following statements are equivalent:*

- 1°.  $f$  is anti-topical.
- 2°. We have (45) and

$$f^{\theta(\varphi)}(y) = f(y)^{-1}, \quad \forall y \in X. \quad (105)$$

- 3°. We have (45) and

$$f^{\theta(\psi)}(y, d) = f(y)^{-1}, \quad \forall x \in X, \forall d \in \mathcal{K}. \quad (106)$$

**Proof.**  $1^\circ \Rightarrow 2^\circ$  : If  $1^\circ$  holds, then by Theorem 15 b) we have (45). Furthermore, by Corollary 18 b) we have (77), and hence  $f^{\theta(\varphi)}(y) \geq f(y)^{-1}$ .

In the opposite direction, if  $y \in X \setminus \{\inf X\}$ , then by (98) and (4), we have

$$f^{\theta(\varphi)}(y) = \inf_{x \in X} \{f(x)^{-1} \dot{\otimes} (x/y)^{-1}\} \leq f(y)^{-1} \dot{\otimes} (y/y)^{-1} = f(y)^{-1},$$

whence, finally,  $f^{\theta(\varphi)}(y) = f(y)^{-1}, \forall y \in X \setminus \{\inf X\}$ . Furthermore, for  $y = \inf X$  we have, by (98) and (45),

$$\begin{aligned} f^{\theta(\varphi)}(\inf X) &= \inf_{x \in X} f(x)^{-1} \dot{\otimes} (x/\inf X)^{-1} \leq f(\inf X)^{-1} \dot{\otimes} (\inf X/\inf X)^{-1} \\ &= \top^{-1} \dot{\otimes} \top^{-1} = \varepsilon \dot{\otimes} \varepsilon = \varepsilon = \top^{-1} = f(\inf X)^{-1}. \end{aligned}$$



$2^\circ \Rightarrow 1^\circ$ . If  $2^\circ$  holds, then for all  $x, y \in X$  we have, by (98) and (105),  $f(x)^{-1} \dot{\otimes} (x/y)^{-1} \geq f^{\theta(\varphi)}(y) = f(y)^{-1}$ . Hence  $f(y) \dot{\otimes} (x/y)^{-1} \geq f(x)$  (by Lemma 11) and therefore, by Theorem 15 b),  $f$  is anti-topical.

$1^\circ \Rightarrow 3^\circ$ . If  $1^\circ$  holds, then by Theorem 15 b) we have (45). Assume now  $1^\circ$  and let  $x, y \in X, d \in \mathcal{K}$ . Then by (56),  $1^\circ$  and Theorem 15 b) we have

$$f(y) \dot{\otimes} \overline{s}_{y,d}(x) = f(y) \dot{\otimes} \sup\{(x/y)^{-1}, d\} \geq f(y) \dot{\otimes} (x/y)^{-1} \geq f(x),$$

and thus  $f(x)^{-1} \dot{\otimes} \overline{s}_{y,d}(x) \geq f(y)^{-1}$  (by Lemma 11). Hence by (99), and since  $x \in X$  was arbitrary, we get

$$f^{\theta(\psi)}(y, d) = \inf_{x \in X} \{f(x)^{-1} \dot{\otimes} \overline{s}_{y,d}(x)\} \geq f(y)^{-1}. \quad (107)$$

In the opposite direction, by (99) and (56), for any  $f : X \rightarrow \overline{\mathcal{K}}, y \in X \setminus \{\inf X\}$  and  $d \in \mathcal{K} \setminus \{\varepsilon\}$  we have

$$\begin{aligned} f^{\theta(\psi)}(y, d) &\leq f(d^{-1}y)^{-1} \dot{\otimes} \overline{s}_{y,d}(d^{-1}y) \\ &= f(d^{-1}y)^{-1} \dot{\otimes} \sup\{(d^{-1}y/y)^{-1}, d\} = f(d^{-1}y)^{-1} \dot{\otimes} d. \end{aligned}$$

Hence, since by  $1^\circ$  and  $d \in \mathcal{K} \setminus \{\varepsilon\}$  we have

$$f(d^{-1}y)^{-1} \dot{\otimes} d = (d \dot{\otimes} f(y))^{-1} \dot{\otimes} d = (d^{-1} \dot{\otimes} f(y)^{-1}) \dot{\otimes} d = f(y)^{-1},$$

we obtain the opposite inequality to (107), and hence the equality (106) for  $y \in X \setminus \{\inf X\}, d \in \mathcal{K} \setminus \{\varepsilon\}$ .

Assume now that  $y = \inf X$  and  $d \in \mathcal{K} \setminus \{\varepsilon\}$ . Then by (21), (45), (8) and (11) we have

$$\begin{aligned} f^{\theta(\psi)}(\inf X, d) &= \inf_{x \in X} f(x)^{-1} \dot{\otimes} \sup\{(x/\inf X)^{-1}, d\} \\ &= \inf_{x \in X} f(x)^{-1} \dot{\otimes} \sup\{\varepsilon, d\} = \inf_{x \in X} f(x)^{-1} \dot{\otimes} d \\ &\leq f(\inf X)^{-1} \dot{\otimes} d = \top^{-1} \dot{\otimes} d = \varepsilon \dot{\otimes} d = \varepsilon d = \varepsilon = f(\inf X)^{-1}, \end{aligned}$$

so we obtain the opposite inequality to (107), and hence the equality (106) for  $y = \inf X, d \in \mathcal{K} \setminus \{\varepsilon\}$ . Thus, the equality (106) holds for all  $y \in X, d \in \mathcal{K} \setminus \{\varepsilon\}$ .

Finally, for  $y \in X, d = \varepsilon$  we have, by (101),  $1^\circ$  and the implication  $1^\circ \Rightarrow 2^\circ$  proved above,

$$f^{\theta(\psi)}(y, \varepsilon) = f^{\theta(\varphi)}(y) = f(y)^{-1}.$$

$3^\circ \Rightarrow 1^\circ$ . If  $3^\circ$  holds, then for all  $x, y \in X, d \in \mathcal{K}$ , we have, by (99) and (106),

$$f(x)^{-1} \dot{\otimes} \overline{s}_{y,d}(x) \geq f^{\theta(\psi)}(y, d) = f(y)^{-1},$$

whence  $f(y) \dot{\otimes} \overline{s}_{y,d}(x) \geq f(x)$  (by Lemma 11). Consequently, by Theorem 15 b),  $f$  is anti-topical.  $\square$

The following Corollary of Theorem 31 gives characterizations of the functions  $f$  that satisfy the equalities (105), (106):

**Corollary 32** *Let  $(X, \mathcal{K})$  be a pair that satisfies  $(A0')$ ,  $(A1)$ . For a function  $f : X \rightarrow \overline{\mathcal{K}}$  the following statements are equivalent:*

- 1°. *We have (105).*
- 2°. *We have (106).*
- 3°. *Either  $f$  is anti-topical or  $f \equiv \varepsilon$ .*

**Proof.**  $1^\circ \Rightarrow 3^\circ$ . Assume that we have  $1^\circ$  and  $f$  is not anti-topical, so  $f(\inf X) \neq \top$  (by Theorem 31). Then by (105) applied to  $y = \inf X$  we have

$$\inf_{x \in X} \{f(x)^{-1} \dot{\otimes} (x / \inf X)^{-1}\} = f^{\theta(\varphi)}(\inf X) = f(\inf X)^{-1},$$

whence, by (21) and since  $f(\inf X)^{-1} > \varepsilon$ , it follows that

$$f(x)^{-1} \dot{\otimes} \varepsilon = f(x)^{-1} \dot{\otimes} (x / \inf X)^{-1} > \varepsilon, \quad \forall x \in X.$$

Therefore we must have  $f(x)^{-1} = \top, \forall x \in X$ , so  $f \equiv \varepsilon$ .

$3^\circ \Rightarrow 1^\circ$ . Assume  $3^\circ$ . If  $f$  is anti-topical, then we have (105) by Theorem 31. On the other hand, if  $f \equiv \varepsilon$ , then (105) holds since by (98) we have

$$\varepsilon^{\theta(\varphi)}(y) = \inf_{x \in X} \{\varepsilon(x)^{-1} \dot{\otimes} (x/y)^{-1}\} = \top = \varepsilon(y)^{-1}, \quad \forall y \in X.$$

$2^\circ \Rightarrow 3^\circ$ . If  $2^\circ$  holds, then by (57) and (106) applied to  $d = \varepsilon$  we have

$$f^{\theta(\varphi)}(y) = f^{\theta(\psi)}(y, \varepsilon) = f(y)^{-1}, \quad \forall y \in X,$$

so we have (105). Hence by the implication  $1^\circ \Rightarrow 3^\circ$  proved above, we obtain  $3^\circ$ .

$3^\circ \Rightarrow 2^\circ$ . Assume  $3^\circ$ . If  $f$  is anti-topical, then we have (106) by Theorem 31. On the other hand, if  $f \equiv \varepsilon$ , then (106) holds since by (99) we have

$$\varepsilon^{\theta(\psi)}(y, d) = \inf_{x \in X} \{\varepsilon(x)^{-1} \dot{\otimes} \overline{s}_{y,d}(x)\} = \top = \varepsilon(y)^{-1}, \quad \forall y \in X, \forall d \in \mathcal{K}. \quad \square$$

Now we shall attempt to apply the second conjugates (i.e., conjugates of conjugates), or briefly, *biconjugates*, of a function  $f : X \rightarrow \overline{\mathcal{K}}$ , for the study of topical and anti-topical functions. In the particular case of the coupling function  $\pi_\mu : R_{\max}^n \times R_{\max}^n \rightarrow R_{\max}$  of (86), in [9] it has been shown (see [9, Theorem 5.4 and Lemma 5.1]) that  $f$  is topical if and only if  $f^{c(\pi_\mu)c(\pi_\mu)} = f$  (see also [12, Theorem 6.3 and formula (6.32)] for an extension from  $R_{\max}^n$  to  $A^n$ ). This approach has used the so-called dual mappings of Moreau (see e.g. [8]), which we shall now try to adapt.

We recall that  $\overline{\mathcal{K}}^X$  denotes the set of all functions  $f : X \rightarrow \overline{\mathcal{K}}$ .

**Definition 33** Let  $(X, \mathcal{K}), (Y, \mathcal{K})$  be two pairs satisfying  $(A0'), (A1)$ .

a) For any coupling function  $\pi : X \times Y \rightarrow \overline{\mathcal{K}}$  the coupling function  $\overline{\pi} : Y \times X \rightarrow \overline{\mathcal{K}}$  defined by

$$\overline{\pi}(y, x) := \pi(x, y), \quad \forall x \in X, \forall y \in Y \quad (108)$$

will be called the *reflexion* of  $\pi$ .

b) The *dual* of any mapping  $u : \overline{\mathcal{K}}^X \rightarrow \overline{\mathcal{K}}^Y$  is the mapping  $u' : \overline{\mathcal{K}}^Y \rightarrow \overline{\mathcal{K}}^X$  defined by

$$h^{u'} := \inf_{\substack{g \in \overline{\mathcal{K}}^X \\ g^u \leq h}} g, \quad \forall h \in \overline{\mathcal{K}}^Y, \quad (109)$$

where we write  $g^u$  and  $h^{u'}$  instead of  $u(g)$  and  $u'(h)$  respectively.

c) The *bidual* of any mapping  $u : \overline{\mathcal{K}}^X \rightarrow \overline{\mathcal{K}}^Y$  is the mapping  $f \rightarrow (f^u)^{u'}$  of  $\overline{\mathcal{K}}^X$  into  $\overline{\mathcal{K}}^X$ .

For the Fenchel-Moreau conjugation  $u = c(\pi)$  (see Definition 21) we have

**Lemma 34** *If  $(X, \mathcal{K}), (Y, \mathcal{K})$  are two pairs satisfying  $(A0'), (A1)$ , and  $\pi : X \times Y \rightarrow \overline{\mathcal{K}}$  is a coupling function, then*

$$c(\pi)' = c(\overline{\pi}). \quad (110)$$

**Proof.** By (78), Lemma 11 and (108), for any  $g \in \overline{\mathcal{K}}^X$  and  $h \in \overline{\mathcal{K}}^Y$  we have the equivalences

$$\begin{aligned} g^{c(\pi)}(y) &\leq h(y), \forall y \in Y \\ \Leftrightarrow g(x)^{-1} \pi(x, y) &\leq h(y), \forall x \in X, \forall y \in Y \\ \Leftrightarrow h(y)^{-1} \pi(x, y) &\leq g(x), \forall x \in X, \forall y \in Y \\ \Leftrightarrow h(y)^{-1} \overline{\pi}(y, x) &\leq g(x), \forall x \in X, \forall y \in Y \\ \Leftrightarrow h^{c(\overline{\pi})}(x) &\leq g(x), \forall x \in X, \end{aligned} \quad (111)$$

whence by (109),

$$h^{c(\pi)'}(x) = \inf_{\substack{g \in \overline{\mathcal{K}}^X \\ g^{c(\pi)} \leq h}} g(x) = \inf_{\substack{g \in \overline{\mathcal{K}}^X \\ h^{c(\overline{\pi})} \leq g}} g(x) = h^{c(\overline{\pi})}(x), \quad \forall x \in X. \quad \square$$

**Remark 35** a) In the particular case where  $X = Y$  and  $\pi = \varphi : X \times X \rightarrow \overline{\mathcal{K}}$  is the coupling function (82), the  $\overline{\varphi}$ -conjugate function  $f^{c(\overline{\varphi})}$  of any function  $f : X \rightarrow \overline{\mathcal{K}}$  such that  $f(\inf X) \neq \top$  (e.g., of any topical function  $f$ ) satisfies

$$\begin{aligned} f^{c(\overline{\varphi})}(y) &= \sup_{x \in X} f(x)^{-1} \overline{\varphi}(x, y) = \sup_{x \in X} f(x)^{-1} \varphi(y, x) = \sup_{x \in X} f(x)^{-1} (y/x) \\ &\geq f(\inf X)^{-1} (y/\inf X) = \top, \quad \forall y \in X, \end{aligned}$$

whence  $f^{c(\overline{\varphi})}(y) = \top, \forall y \in X$ , so  $f^{c(\overline{\varphi})}$  is anti-topical, while for  $f \equiv \top$  we have

$$\top^{c(\overline{\varphi})}(y) = \sup_{x \in X} \top(x)^{-1} (y/x) = \sup_{x \in X} \varepsilon y/x = \varepsilon, \quad \forall y \in X,$$

so  $\top^{c(\overline{\varphi})}$  is topical. This should be compared with the facts that for any function  $f : X \rightarrow \overline{\mathcal{K}}, f^{c(\varphi)}$  is anti-topical (see Remark 24 a)) and for any function  $f :$

$X \rightarrow \overline{\mathcal{K}}$  such that  $f \not\equiv \varepsilon$ ,  $f^{\theta(\varphi)}$  is topical, while for  $f \equiv \varepsilon$ ,  $\varepsilon^{\theta(\varphi)}$  is anti-topical (see Remark 30a)).

b) The inequalities occurring in (111) are equivalent to each of

$$\pi(x, y) \leq h(y)g(x), \quad \overline{\pi}(y, x) \leq g(x)h(y), \quad \forall x \in X, \forall y \in Y, \quad (112)$$

which might be called “generalized Fenchel-Young inequalities”, because of the particular case of the so-called “natural coupling function”  $\pi : X \times Y \rightarrow R$  defined by  $\pi(x, y) := xy$ ,  $\forall x \in R, \forall y \in R$ .

In the particular case where  $X = Y$  and  $\pi : X \times X \rightarrow \overline{\mathcal{K}}$  is a symmetric coupling function, that is,

$$\pi(x, y) = \pi(y, x), \quad \forall x \in X, \forall y \in X, \quad (113)$$

Lemma 34 reduces to the following:

**Corollary 36** *If  $(X, \mathcal{K})$  is a pair satisfying  $(A0')$ ,  $(A1)$ , and  $\pi : X \times X \rightarrow \overline{\mathcal{K}}$  is a symmetric coupling function, then  $c(\pi)$  is “self-dual”, that is,*

$$c(\pi)' = c(\pi). \quad (114)$$

**Remark 37** In particular, for  $\mathcal{K} = R_{\max}$  and  $X$  an arbitrary set, Corollary 36 has been obtained in [9, Lemma 5.1].

For the Fenchel-Moreau biconjugates  $f^{c(\varphi)c(\varphi)'} := (f^{c(\varphi)})^{c(\varphi)'}$  with respect to the coupling function  $\varphi$  of (82) we obtain

**Theorem 38** *If  $(X, \mathcal{K})$  is a pair satisfying  $(A0')$ ,  $(A1)$ , then for every function  $f : X \rightarrow \overline{\mathcal{K}}$  we have*

$$f^{c(\varphi)c(\varphi)'} \leq f. \quad (115)$$

*If  $f : X \rightarrow \overline{\mathcal{K}}$  is topical, then*

$$f^{c(\varphi)c(\varphi)'} = f; \quad (116)$$

*the converse is not true, since (116) is also satisfied e.g. for the anti-topical function  $f \equiv \top$ .*

**Proof.** For any function  $f : X \rightarrow \overline{\mathcal{K}}$ , applying formula (109) to  $h = f^{c(\varphi)}$  and  $u = c(\varphi)$  we obtain

$$f^{c(\varphi)c(\varphi)'}(x) := \inf_{\substack{g \in \overline{\mathcal{K}}^X \\ g^{c(\varphi)} \leq f^{c(\varphi)}}} g(x) \leq f(x), \quad \forall x \in X,$$

which proves (115).

Furthermore, observe that for any function  $f : X \rightarrow \overline{\mathcal{K}}$  we have, using (110) with  $\pi = \varphi$  of (82),

$$f^{c(\varphi)c(\varphi)'}(x) = (f^{c(\varphi)})^{c(\overline{\varphi})}(x), \quad \forall x \in X. \quad (117)$$

Now let  $f : X \rightarrow \overline{\mathcal{K}}$  be a topical function and let  $h = f^{c(\varphi)}$  and  $x \in X$ . Then by (108) for  $\pi = \varphi$  of (82) and by  $f^{c(\varphi)}(y) = f(y)^{-1}, \forall y \in X$  (see Theorem 25), we have

$$\begin{aligned} f^{c(\varphi)c(\overline{\varphi})}(x) &= h^{c(\overline{\varphi})}(x) = \sup_{y \in X} h(y)^{-1} \overline{\varphi}(y, x) = \sup_{y \in X} h(y)^{-1}(y/x) \\ &= \sup_{y \in X} (f^{c(\varphi)}(y)^{-1}(y/x)) = \sup_{y \in X} f(y)(y/x) \geq f(x)(x/x). \end{aligned} \quad (118)$$

If  $x \neq \inf X$ , then  $f(x)x/x = f(x)$ , and hence by (118),

$$f^{c(\varphi)c(\overline{\varphi})}(x) \geq f(x)x/x = f(x).$$

On the other hand, if  $x = \inf X$ , then  $f(\inf X) = \varepsilon$  (since  $f$  is topical), and hence by (118),

$$f^{c(\varphi)c(\overline{\varphi})}(\inf X) \geq f(\inf X)(\inf X/\inf X) = f(\inf X)\top = \varepsilon = f(\inf X).$$

Thus for any topical function  $f : X \rightarrow \overline{\mathcal{K}}$  we have  $f^{c(\varphi)c(\overline{\varphi})}(x) \geq f(x)$ ,  $\forall x \in X$ , which together with (115) yields the equality (116).

Finally, let us show that the converse implication is not true. Indeed, by Remarks 22a) and 35a) we have

$$\top^{c(\varphi)c(\overline{\varphi})} = (\top^{c(\varphi)})^{c(\overline{\varphi})} = \varepsilon^{c(\overline{\varphi})} = \top,$$

whence (116) for  $f \equiv \top$ , which, since  $f \equiv \top$  is anti-topical (see Remark 20), completes the proof.  $\square$

**Remark 39** In [9, Theorem 5.4] it has been shown that for any topical function  $f : R_{\max}^n \rightarrow \overline{R}$  the equality (116) holds for  $\varphi$  replaced by the additive min-type coupling function  $\pi_\mu$  of (86) and conversely, every function  $f : R_{\max}^n \rightarrow \overline{R}$  such that  $f^{c(\pi_\mu)c(\pi_\mu)'} = f$  is topical. The proof of [9] has used the fact that  $\pi_\mu$  is symmetric and hence self-dual, but this is not the case for  $\varphi$ ; furthermore,  $f^{c(\pi_\mu)}$  is topical for every function  $f : R_{\max}^n \rightarrow \overline{R}$ , but  $f^{c(\varphi)}$  is anti-topical for every function  $f : X \rightarrow \overline{\mathcal{K}}$  (see Remark 24a)), and therefore the arguments of the proof of [9, Theorem 5.4] do not work for the case of  $f^{c(\varphi)c(\varphi)'}$ .

Another way of characterizing topical and anti-topical functions with the aid of biconjugates is to combine conjugates with lower conjugates as follows:

**Theorem 40** *If  $(X, \mathcal{K})$  is a pair satisfying (A0'), (A1), then:*

a) *A function  $f : X \rightarrow \overline{\mathcal{K}}$  is topical if and only if  $f \not\equiv \top$  and*

$$f^{c(\varphi)\theta(\varphi)} = f. \quad (119)$$

b) *A function  $f : X \rightarrow \overline{\mathcal{K}}$  is anti-topical if and only if*

$$f^{\theta(\varphi)c(\varphi)} = f. \quad (120)$$

**Proof.** a) If  $f : X \rightarrow \overline{\mathcal{K}}$  is topical, then  $f \not\equiv \top$  (by (38)). Furthermore,  $f^{c(\varphi)}$  is anti-topical (by Remark 24a)), and hence by Theorem 31 applied to the function  $f^{c(\varphi)}$ , and Theorem 25, we obtain

$$(f^{c(\varphi)})^{\theta(\varphi)}(y) = f^{c(\varphi)}(y)^{-1} = (f(y)^{-1})^{-1} = f(y), \quad \forall y \in X,$$

so  $f$  satisfies (119).

Conversely, assume that  $f$  satisfies  $f \not\equiv \top$  and (119). Then  $f^{c(\varphi)} \not\equiv \varepsilon$ , since otherwise by (119) and (100) we would obtain  $f(y) = (f^{c(\varphi)})^{\theta(\varphi)}(y) = \varepsilon^{\theta(\varphi)}(y) = \top(y), \forall y \in X$ , in contradiction with the assumption  $f \not\equiv \top$ . Consequently, by Remark 30a),  $f = (f^{c(\varphi)})^{\theta(\varphi)}$  is topical.

b) If  $f : X \rightarrow \overline{\mathcal{K}}$  is anti-topical, then  $f \not\equiv \varepsilon$  (by (45)). Consequently, by Remark 30a),  $f^{\theta(\varphi)}$  is topical and hence, by Theorem 25 applied to  $f^{\theta(\varphi)}$ , and Theorem 31 applied to  $f$ , we obtain

$$f^{\theta(\varphi)c(\varphi)}(y) = f^{\theta(\varphi)}(y)^{-1} = (f(y)^{-1})^{-1} = f(y), \quad \forall y \in X.$$

Conversely, if  $f$  satisfies (120), then by Remark 24 a),  $f$  is anti-topical.

□

## 5 Polars of a set

We shall study the following concept of “polar set” in our framework:

**Definition 41** Let  $(X, \mathcal{K})$  be a pair satisfying  $(A0')$ ,  $(A1)$ , let  $\mathcal{K}$  be enlarged to  $\overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$  as above, and let  $\pi : X \times Y \rightarrow \overline{\mathcal{K}}$  be a coupling function. The  $\pi$ -polar of a subset  $G$  of  $X$  is the subset  $G^{o(\pi)}$  of  $Y$  defined by

$$G^{o(\pi)} := \{y \in Y \mid \pi(g, y) \leq e, \forall g \in G\} = \{y \in Y \mid \sup_{g \in G} \pi(g, y) \leq e\}. \quad (121)$$

In particular, for the coupling function  $\pi = \varphi : X \times X \rightarrow \overline{\mathcal{K}}$  of (82) we have the  $\varphi$ -polar of  $G$  defined by

$$G^o := G^{o(\varphi)} := \{y \in Y \mid g/y \leq e, \forall g \in G\} = \{y \in Y \mid \sup_{g \in G} (g/y) \leq e\}, \quad (122)$$

which we shall call simply the *polar* of  $G$ .

**Remark 42** a) In particular, for  $Y = X, \pi : X \times Y \rightarrow \overline{\mathcal{K}}$  of (82) and any (non-empty) set  $G \subseteq X$ , by (122) and (21) we have  $\inf X \notin G^o$ .

b) Generalizing a concept of [9, 12], given two pairs  $(X, \mathcal{K}), (Y, \mathcal{K})$  satisfying assumptions  $(A0')$  and  $(A1)$ , a coupling function  $\pi : X \times Y \rightarrow \overline{\mathcal{K}}$ , where  $\overline{\mathcal{K}}$  is the minimal enlargement of  $\mathcal{K}$ , and a subset  $G$  of  $X$ , we shall call  $\pi$ -support function of  $G$  the function  $\sigma_{G, \pi} : Y \rightarrow \overline{\mathcal{K}}$  defined by

$$\sigma_{G, \pi}(y) := \sup_{g \in G} \pi(g, y), \quad \forall y \in Y. \quad (123)$$

In the sequel we shall consider the particular case when  $Y = X$  and  $\pi = \varphi : X \times X \rightarrow \overline{\mathcal{K}}$  of (82). In this case we shall call

$$\sigma_G(y) := \sigma_{G,\varphi}(y) = \sup_{g \in G} (g/y), \quad \forall y \in X, \quad (124)$$

the *support function* of  $G$ . Note that for any set  $G \subseteq X$ ,  $\sigma_G$  is an *anti-topical function* on  $X$ ; indeed, for each  $g \in G$ , the function  $y \rightarrow g/y = \varphi(g, y)$ ,  $\forall y \in X$ , is anti-topical (by Lemma 27b)) and the supremum of any family of anti-topical functions is anti-topical. Note also that for  $y = \inf X$  we have

$$\sigma_G(\inf X) = \sup_{g \in G} (g/\inf X) = \top, \quad (125)$$

and that the polar of  $G$  is the “ $e$ -level set” of  $\sigma_G$ , that is,

$$G^o = \{y \in X \mid \sigma_G(y) \leq e\}. \quad (126)$$

c) We recall that a subset  $G$  of  $X$  is said to be *downward*, if

$$g \in G, x \in X, x \leq g \Rightarrow x \in G, \quad (127)$$

and *upward* if

$$g \in G, x \in X, x \geq g \Rightarrow x \in G. \quad (128)$$

Let us observe that for any set  $G$  the polar  $G^o = G^{o(\varphi)}$  is an *upward set*; indeed, if  $y \in G^o, y' \in X, y \leq y'$ , then  $y' \in X \setminus \{\inf X\}$  and  $g/y' \leq g/y \leq e$ , so  $y' \in G^o$ . Thus  $G^o$  is upward. On the other hand, for the reflexion  $\overline{\varphi}$  of  $\varphi$  (see (108)), the  $\overline{\varphi}$ -polar set

$$G^{o(\overline{\varphi})} := \{y \in X \mid y/g \leq e, \forall g \in G\} \quad (129)$$

is *downward*; indeed, if  $y \in G^{o(\overline{\varphi})}, y' \leq y$ , then  $y'/g \leq y/g \leq e, \forall g \in G$ .

We recall (see e.g. [11] and the references therein) that for two sets  $X$  and  $Y$  the *dual* of any mapping  $\Delta : 2^X \rightarrow 2^Y$  (where  $2^X$  denotes the family of all subsets of  $X$ ) is the mapping  $\Delta' : 2^Y \rightarrow 2^X$  defined by

$$\Delta'(P) := \{x \in X \mid P \subseteq \Delta(\{x\})\}, \quad \forall P \subseteq Y, \quad (130)$$

Furthermore, we recall that a mapping  $\Delta : 2^X \rightarrow 2^Y$  is called a *polarity* (in [11], [10] and earlier references we have used the term *duality*) if

$$\Delta(G) = \cap_{g \in G} \Delta(\{g\}), \quad \forall G \subseteq X. \quad (131)$$

It is well-known and immediate that the dual  $\Delta'$  of any mapping  $\Delta : 2^X \rightarrow 2^Y$  is a polarity.

**Definition 43** If  $X, Y$  are two sets and  $\pi : X \times Y \rightarrow \overline{\mathcal{K}}$  is a coupling function, the *polarity associated to  $\pi$* , or briefly, the  $\pi$ -*polarity*, is the mapping  $\Delta_\pi : 2^X \rightarrow 2^Y$  defined by

$$\Delta_\pi(G) := G^{o(\pi)}, \quad \forall G \subseteq X; \quad (132)$$

with  $G^{o(\pi)}$  of (121); the mapping  $\Delta = \Delta_\pi$  satisfies (131), so it is indeed a polarity.

We have the following extension of [11, Lemma 2.1]:

**Lemma 44** *For the dual  $\Delta'_\pi$  of the mapping  $\Delta = \Delta_\pi$  we have*

$$\Delta'_\pi = \Delta_{\bar{\pi}}, \quad (133)$$

where  $\bar{\pi}$  is the reflexion (108) of  $\pi$ .

**Proof.** By (130), (132), (121), (108) and (132) (applied to  $\bar{\pi}$  and  $P$ ), for any subset  $P$  of  $Y$  we have

$$\begin{aligned} \Delta'_\pi(P) &= \{x \in X \mid P \subseteq \Delta_\pi(\{x\})\} = \{x \in X \mid \pi(x, y) \leq e, \forall y \in P\} \\ &= \{x \in X \mid \bar{\pi}(y, x) \leq e, \forall y \in P\} = \Delta_{\bar{\pi}}(P). \quad \square \end{aligned}$$

In the particular case when  $\pi$  is a symmetric coupling function, that is, satisfies (113), Lemma 44 reduces to the following:

**Corollary 45** *If  $(X, \mathcal{K})$  is a pair satisfying (A0'), (A1), and  $\pi : X \times X \rightarrow \bar{\mathcal{K}}$  is a symmetric coupling function, then  $\Delta_\pi : 2^X \rightarrow 2^X$  is self-dual, that is,*

$$\Delta'_\pi = \Delta_\pi. \quad (134)$$

**Remark 46** In particular, for  $\mathcal{K} = R_{\max}$  and  $X$  an arbitrary set, Corollary 45 has been obtained in [11, Lemma 2.1].

If  $\Delta : 2^X \rightarrow 2^Y$  is a polarity, with dual  $\Delta'$ , then by definition

$$\Delta' \Delta(G) := \Delta'(\Delta(G)), \quad \forall G \subseteq X, \quad (135)$$

and a subset  $G$  of  $X$  is called  $\Delta' \Delta$ -convex if  $G = \Delta' \Delta(G)$ , or equivalently (see [10]), if for each  $x \notin G$  there exists  $y \in Y$  such that

$$G \subseteq \Delta'(\{y\}), \quad x \in X \setminus \Delta'(\{y\}). \quad (136)$$

In other words, a subset  $G$  of  $X$  is  $\Delta' \Delta$ -convex if and only if it is  $\mathcal{M}$ -convex, i.e., “convex with respect to the family of sets”  $\mathcal{M} := \{\Delta'(\{y\}) \mid y \in Y\} \subseteq X$  (see e.g. [10] and the references therein).

For the  $\pi$ -polarity  $\Delta_\pi : 2^X \rightarrow 2^Y$  of (132),  $\Delta'_\pi \Delta_\pi$ -convexity of a set  $G \subseteq X$  means that for each  $x \notin G$  there should exist  $y \in Y$  such that  $y \in \Delta_\pi(G) \setminus \Delta_\pi(\{x\})$ , or in other words such that

$$\sup_{g \in G} \pi(g, y) \leq e, \quad \pi(x, y) \not\leq e. \quad (137)$$

For  $Y = X$  and the coupling function  $\pi = \varphi$  of (82),  $\Delta'_\varphi \Delta_\varphi$ -convexity of a set  $G \subseteq X$  means that for each  $x \notin G$  there should exist  $y \in X$ , or what is equivalent (by Remark 42a)),  $y \in X \setminus \{\inf X\}$ , such that

$$\sup_{g \in G} (g/y) \leq e, \quad x/y \not\leq e. \quad (138)$$



In the particular case when  $\leq$  is a total order on  $\mathcal{K}$ , (137) and (138) become, respectively,

$$\sup_{g \in G} \pi(g, y) \leq e < \pi(x, y), \quad (139)$$

and

$$\sup_{g \in G} (g/y) \leq e < x/y. \quad (140)$$

Let us consider now the counterpart for “bipolars”  $\Delta'_\varphi \Delta_\varphi(G)$  of the problem mentioned in Remark 39 (namely, to study the functions  $f : X \rightarrow \overline{\mathcal{K}}$  such that  $f^{c(\varphi)c(\varphi)'} = f$ ). Some results on  $\Delta'_\varphi \Delta_\varphi$ -convex sets  $G \subseteq X$ , without using this terminology, have been given in [13]. For example, as in [13], let us recall the following conditions on the relations between the topologies of  $\mathcal{K}$  and  $X$  (for various other possible conditions see [2] and the references therein):

(A2) For each  $x \in X$  the function  $u_x : \lambda \in \mathcal{K} \rightarrow \lambda x \in X$  is continuous, that is, for any  $x \in X, \lambda \in \mathcal{K}$  and any net  $\{\lambda_\kappa\} \subset \mathcal{K}$  such that  $\lambda_\kappa \rightarrow \lambda$  we have  $\lambda_\kappa x \rightarrow \lambda x$  (here we denote convergence both in  $\mathcal{K}$  and in  $X$  by  $\rightarrow$ , which will lead to no confusion).

(A3) For each  $y \in X \setminus \{\inf X\}$  the function  $y^\diamond(\cdot) = \cdot/y : X \rightarrow \mathcal{K}$  is continuous.

If (A2) holds, then a set  $G \subseteq X$  is said to be *closed along rays* if for each  $x \in X$  the set

$$H_x := \{\lambda \in \mathcal{K} \mid \lambda x \in G\} \quad (141)$$

is closed in  $\mathcal{K}$  (that is, for any  $x \in X$  and any net  $\{\lambda_\kappa\} \subset \mathcal{K}$  with  $\{\lambda_\kappa x\} \subset G$  such that  $\lambda_\kappa \rightarrow \lambda \in \mathcal{K}$  we have  $\lambda x \in G$ ).

We can recall now the following result of [13, Theorem 10]:

**Theorem 47** *Let  $(X, \mathcal{K})$  be a pair satisfying (A0'), (A1), (A2) and (A3). For a subset  $G$  of  $X$  let us consider the following statements:*

- 1°.  $G$  is a closed downward set.
- 2°.  $G$  is closed along rays and downward.
- 3°. We have  $\Delta'_\varphi \Delta_\varphi(G) = G$ , that is,  $G$  is  $\Delta'_\varphi \Delta_\varphi$ -convex, where  $\varphi$  is the coupling function (82).

*Then we have the implications  $1^\circ \Rightarrow 2^\circ \Rightarrow 3^\circ$ .*

*If the canonical order  $\leq$  on  $\mathcal{K}$  is total and if (A3) holds, then the statements  $1^\circ$ ,  $2^\circ$ , and  $3^\circ$  are equivalent to each other and to the following statements:*

- 4°. For each  $x \in (X \setminus G) \setminus \{\inf X\}$  there exists  $y \in X \setminus \{\inf X\}$  satisfying (140).
- 5°. For each  $x \in (X \setminus G) \setminus \{\inf X\}$  there exists  $y \in X \setminus \{\inf X\}$  such that

$$\sup_{g \in G} g/y < x/y. \quad (142)$$

In particular, if  $X = \mathcal{K}^n$ , where  $\mathcal{K} = R_{\max}$ , then the canonical order  $\leq$  and the lattice operations in  $\mathcal{K}$  and  $X$  coincide with the usual order and lattice operations on  $\mathcal{K}^n$  and conditions (A2) and (A3) are satisfied for the order topology (by [5], Corollary 2.11), which is known to coincide with the usual topologies on  $\mathcal{K}$  and  $X$  (see [5], the last observation before Proposition 2.9).

## 6 Support set. Support set at a point

For a pair  $(X, \mathcal{K})$  satisfying (A0), (A1'), we shall denote by  $\tilde{\mathcal{T}}$  the set of all *elementary topical* functions  $\tilde{t} = t_y : X \rightarrow \mathcal{K}$ , that is, all functions of the form

$$\tilde{t}(x) = t_y(x) := x/y, \quad \forall x \in X, \forall y \in X \setminus \{\inf X\}; \quad (143)$$

we use here the notation  $\tilde{t}_y$  in order to avoid confusion with the functions  $t_y : X \rightarrow \bar{\mathcal{K}}$  of (32), (34).

We recall that for a set  $X$ , a coupling function  $\pi : X \times X \rightarrow R_{\max}$ , respectively  $\pi : X \times X \rightarrow A$ , where  $A = (A, \oplus, \otimes)$  is a conditionally complete lattice ordered group, and a function  $f : X \rightarrow \bar{R}$ , respectively  $f : X \rightarrow \bar{A}$ , the  $\pi$ -*support set* of  $f$  is the subset of  $X$  defined [9, 12] by

$$\text{Supp}(f, \pi) := \{y \in X \mid \pi(x, y) \leq f(x), \forall x \in X\}. \quad (144)$$

More generally, for a pair  $(X, \mathcal{K})$  satisfying (A0), (A1') and the coupling function  $\pi = \varphi : X \times X \rightarrow \bar{\mathcal{K}}$  of (82) this leads us to define the following concept of “support set” that will be suitable in our framework:

**Definition 48** Let  $(X, \mathcal{K})$  be a pair satisfying (A0), (A1'). For a function  $f : X \rightarrow \bar{\mathcal{K}}$  we shall call *support set of  $f$*  (with respect to  $\tilde{\mathcal{T}}$  of (143)) the subset of  $X \setminus \{\inf X\}$  defined by

$$\text{Supp}(f, \tilde{\mathcal{T}}) := \{y \in X \setminus \{\inf X\} \mid \tilde{t}_y \leq f\} = \{y \in X \setminus \{\inf X\} \mid x/y \leq f(x), \forall x \in X\}. \quad (145)$$

**Remark 49** a) For any function  $f : X \rightarrow \bar{\mathcal{K}}$ , the support set  $\text{Supp}(f, \tilde{\mathcal{T}})$  is upward. Indeed, if  $y \in \text{Supp}(f, \tilde{\mathcal{T}})$ ,  $y \leq y'$ , then  $y' \in X \setminus \{\inf X\}$  and

$$\tilde{t}_{y'}(x) = x/y' \leq x/y = \tilde{t}_y(x) \leq f(x), \quad \forall x \in X,$$

so  $y' \in \text{Supp}(f, \tilde{\mathcal{T}})$ .

b) If a function  $f : X \rightarrow \bar{\mathcal{K}}$  is topical then

$$f(x) = \max_{y \in \text{Supp}(f, \tilde{\mathcal{T}})} \tilde{t}_y(x), \quad \forall x \in X. \quad (146)$$

Indeed, by (145) we have the inequality  $\geq$  in (146); furthermore, for any  $x \in X$ , the equality in (146) is attained for  $y = f(x)^{-1}x \in X \setminus f^{-1}(\varepsilon)$  when  $f(x) \in \mathcal{K} \setminus \{\varepsilon\}$ , because then  $x \in X \setminus \{\inf X\}$  (by the topicality of  $f$  and (38)) and

$$\tilde{t}_y(x) = \tilde{t}_{f(x)^{-1}x}(x) = x/f(x)^{-1}x = f(x)x/x = f(x),$$

and for any  $y \in \text{Supp}(f, \tilde{\mathcal{T}})$  when  $f(x) = \varepsilon$ , because then  $\tilde{t}_y(x) = x/y \leq f(x) = \varepsilon$  implies  $\tilde{t}_y(x) = \varepsilon = f(x)$ .

From (146) it follows that for two topical functions  $f_1, f_2 : X \rightarrow \bar{\mathcal{K}}$  we have

$$f_1 = f_2 \Leftrightarrow \text{Supp}(f_1, \tilde{\mathcal{T}}) = \text{Supp}(f_2, \tilde{\mathcal{T}}), \quad (147)$$

which shows that for a topical function  $f : X \rightarrow \mathcal{K}$  the support set  $\text{Supp}(f, \tilde{\mathcal{T}})$  determines uniquely  $f$ . This should be compared with [12, Proposition 7.1].

Generalizing another concept studied in [9, 12], we introduce:

**Definition 50** Let  $(X, \mathcal{K})$  be a pair satisfying (A0), (A1'),  $\pi : X \times Y \rightarrow \overline{\mathcal{K}}$  a coupling function and  $x_0 \in X$ . For a function  $f : X \rightarrow \overline{\mathcal{K}}$  we shall call *support set of  $f$  at  $x_0$*  (with respect to  $\tilde{\mathcal{T}}$  of (143)) the subset of  $X \setminus \{\inf X\}$  defined by

$$\text{Supp}(f, \tilde{\mathcal{T}}; x_0) := \{y \in \text{Supp}(f, \tilde{\mathcal{T}}) | \pi(x_0, y) = f(x_0)\}, \quad (148)$$

that is,

$$\text{Supp}(f, \tilde{\mathcal{T}}; x_0) = \{y \in X \setminus \{\inf X\} | \pi(x, y) \leq f(x), \forall x \in X, \pi(x_0, y) = f(x_0)\}. \quad (149)$$

For a pair  $(X, \mathcal{K})$  satisfying (A0), (A1') and the coupling functions  $\pi = \varphi : X \times X \rightarrow \overline{\mathcal{K}}$  and  $\psi : X \times (X \times \overline{\mathcal{K}}) \rightarrow \overline{\mathcal{K}}$  of (82) and (83) respectively, this leads us to define two concepts of “support set at a point” that will be suitable in our framework:  $\text{Supp}_X(f, \tilde{\mathcal{T}}; x_0)$  and  $\text{Supp}_{(X, \mathcal{K})}(f, \tilde{\mathcal{T}}; x_0)$ .

**Definition 51** Let  $f : X \rightarrow \overline{\mathcal{K}}$  be a function and let  $x_0 \in X$ . We shall call *X-support set of  $f$  at  $x_0$*  the set

$$\text{Supp}_X(f, \tilde{\mathcal{T}}; x_0) := \{y \in X \setminus \{\inf X\} | \tilde{t}_y \leq f, \tilde{t}_y(x_0) = f(x_0)\}, \quad (150)$$

where  $\tilde{t}_y : X \rightarrow \overline{\mathcal{K}}$  are the elementary topical functions (143). In other words:

$$\text{Supp}_X(f, \tilde{\mathcal{T}}; x_0) = \{y \in X \setminus \{\inf X\} | x/y \leq f(x), \forall x \in X, x_0/y = f(x_0)\}. \quad (151)$$

**Proposition 52** Let  $f : X \rightarrow \overline{\mathcal{K}}$  be a topical function and let  $x_0 \in X$  be such that  $f(x_0) \in \mathcal{K} \setminus \{\varepsilon\}$ . Then

$$x_0 f(x_0)^{-1} \in \text{Supp}_X(f, \tilde{\mathcal{T}}; x_0) \neq \emptyset. \quad (152)$$

**Proof.** Since  $f(x_0) \in \mathcal{K} \setminus \{\varepsilon\}$ , by (5), the topicality of  $f$  and Lemma 14a) we have

$$x/f(x_0)^{-1}x_0 = f(x_0)x/x_0 \leq f(x), \quad \forall x \in X. \quad (153)$$

Hence defining

$$y_0 := f(x_0)^{-1}x_0, \quad (154)$$

we obtain  $f(y_0) = f(f(x_0)^{-1}x_0) = e \neq \varepsilon$ , so  $y_0 \in X \setminus f^{-1}(\varepsilon) \subseteq X \setminus \{\inf X\}$  (by (38)) and

$$x/y_0 \leq f(x), \forall x \in X, \quad x_0/y_0 = x_0/f(x_0)^{-1}x_0 = f(x_0),$$

that is,  $y_0 \in \text{Supp}_X(f, \tilde{\mathcal{T}}; x_0) \neq \emptyset$ .  $\square$

**Theorem 53** Let  $f : X \rightarrow \overline{\mathcal{K}}$  be a topical function and let  $x_0 \in X$  be such that  $f(x_0) \in \mathcal{K} \setminus \{\varepsilon\}$ . For an element  $y \in X \setminus \{\inf X\}$  the following statements are equivalent:

- 1°.  $y \in \text{Supp}_X(f, \tilde{\mathcal{T}}; x_0)$ .

2°. We have

$$f(y) = e, \tilde{t}_y(x_0) = f(x_0). \quad (155)$$

3°. We have

$$f(y) = e, \tilde{t}_y(x_0) \geq f(x_0). \quad (156)$$

**Proof.**  $1^\circ \Rightarrow 2^\circ$ . Assume that  $y \in \text{Supp}_X(f, \tilde{\mathcal{T}}; x_0)$ . Then by (150) and (3) we have

$$yf(x_0) = y\tilde{t}_y(x_0) = y(x_0/y) \leq x_0. \quad (157)$$

Hence, since  $f$  is topical, we obtain

$$f(y)f(x_0) = f(yf(x_0)) \leq f(x_0). \quad (158)$$

But, by our assumption we have  $f(x_0) \in \mathcal{K} \setminus \{\varepsilon\}$ , whence multiplying (158) with  $f(x_0)^{-1}$ , we obtain

$$f(y) \leq e. \quad (159)$$

On the other hand, using the fact that by (4) we have

$$\tilde{t}_y(y) = y/y = e, \quad \forall y \in X \setminus \{\inf X\},$$

and again that  $y \in \text{Supp}_X(f, \tilde{\mathcal{T}}; x_0)$ , one has:

$$e = \tilde{t}_y(y) \leq f(y). \quad (160)$$

Combining (159) and (160) yields  $f(y) = e$ .

Finally, from  $y \in \text{Supp}_X(f, \tilde{\mathcal{T}}; x_0)$  and  $f(y) = e$  it follows that we have  $2^\circ$ .

The implication  $2^\circ \Rightarrow 3^\circ$  is obvious.

$3^\circ \Rightarrow 1^\circ$ . Assume that  $y \in X \setminus \{\inf X\}$  satisfies  $3^\circ$ . Then using  $f(y) = e$ , the topicality of  $f$  and Lemma 14a), one has

$$\tilde{t}_y(x) = e(x/y) = f(y)(x/y) \leq f(x), \quad x \in X. \quad (161)$$

On the other hand, by  $3^\circ$  we have

$$\tilde{t}_y(x_0) = x_0/y \geq f(x_0), \quad (162)$$

and hence, by (161) (for  $x = x_0$ ) and (162), we obtain

$$\tilde{t}_y(x_0) = x_0/y = f(x_0). \quad (163)$$

From (163) and (161) it follows that  $y \in \text{Supp}_X(f, \tilde{\mathcal{T}}; x_0)$ .  $\square$

**Definition 54** Let  $f : X \rightarrow \overline{\mathcal{K}}$  be a function and let  $x_0 \in X$ . We shall call  $(X, \mathcal{K})$ -support set of  $f$  at  $x_0$  the set

$$\text{Supp}_{(X, \mathcal{K})}(f, \tilde{\mathcal{T}}; x_0) := \{(y, d) \in (X \setminus \{\inf X\}) \times \mathcal{K} \mid s_{y,d} \leq f, s_{y,d}(x_0) = f(x_0)\}, \quad (164)$$

where  $s_{y,d} : X \rightarrow \overline{\mathcal{K}}$  are the functions (53), (54), (55).

**Proposition 55** *Let  $f : X \rightarrow \overline{\mathcal{K}}$  be a topical function and let  $x_0 \in X$  be such that  $f(x_0) \in \mathcal{K} \setminus \{\varepsilon\}$ . Then*

$$(f(x_0)^{-1}x_0, f(x_0)) \in \text{Supp}_{(X, \mathcal{K})}(f, \tilde{\mathcal{T}}; x_0) \neq \emptyset. \quad (165)$$

**Proof.** If  $f$  is topical and  $f(x_0) \in \mathcal{K} \setminus \{\varepsilon\}$ , then  $(f(x_0)^{-1}x_0, f(x_0)) \in (X \setminus \{\inf X\}) \times (\mathcal{K} \setminus \{\varepsilon\})$  and by (5), the topicality of  $f$  and Theorem 15a) we have

$$\begin{aligned} \inf\{x/f(x_0)^{-1}x_0, d\} &= \inf\{f(x_0)x/x_0, d\} \leq \inf\{f(x), d\} \\ &\leq f(x), \quad \forall x \in X, \forall d \in \mathcal{K}. \end{aligned} \quad (166)$$

Consequently,

$$\begin{aligned} s_{f(x_0)^{-1}x_0, f(x_0)}(x) &= \inf\{x/f(x_0)^{-1}x_0, f(x_0)\} \leq f(x), \quad \forall x \in X, \\ s_{f(x_0)^{-1}x_0, f(x_0)}(x_0) &= \inf\{x_0/f(x_0)^{-1}x_0, f(x_0)\} = f(x_0), \end{aligned}$$

that is,  $(f(x_0)^{-1}x_0, f(x_0)) \in \text{Supp}_{(X, \mathcal{K})}(f, \tilde{\mathcal{T}}; x_0) \neq \emptyset$ .  $\square$

**Theorem 56** *Let  $f : X \rightarrow \overline{\mathcal{K}}$  be a topical function and let  $x_0 \in X$  be such that  $f(x_0) \in \mathcal{K} \setminus \{\varepsilon\}$ . For a pair  $(y, d) \in (X \setminus \{\inf X\}) \times \mathcal{K}$ , the following statements are equivalent:*

- 1°.  $(y, d) \in \text{Supp}_{(X, \mathcal{K})}(f, \tilde{\mathcal{T}}; x_0)$ .
- 2°. We have

$$f(y) = e, s_{y, d}(x_0) = f(x_0). \quad (167)$$

- 3°. We have

$$f(y) = e, s_{y, d}(x_0) \geq f(x_0). \quad (168)$$

**Proof.**  $1^\circ \Rightarrow 2^\circ$ . Assume that  $(y, d) \in \text{Supp}_{(X, \mathcal{K})}(f, \tilde{\mathcal{T}}; x_0)$ . Then by (164) we have

$$f(x_0) = s_{y, d}(x_0) = \inf\{x_0/y, d\} \leq x_0/y, \quad (169)$$

whence by (3),

$$yf(x_0) \leq y(x_0/y) \leq x_0.$$

Hence, since  $f$  is topical, we obtain

$$f(y)f(x_0) = f(yf(x_0)) \leq f(x_0). \quad (170)$$

But, by our assumption we have  $f(x_0) \in \mathcal{K} \setminus \{\varepsilon\}$ , whence multiplying (170) with  $f(x_0)^{-1}$  we obtain

$$f(y) \leq e. \quad (171)$$

On the other hand, using again that  $(y, d) \in \text{Supp}_{(X, \mathcal{K})}(f, \tilde{\mathcal{T}}; x_0)$ , we have

$$\inf\{x/y, d\} = s_{y, d}(x) \leq f(x), \quad \forall x \in X,$$

which for  $x = dy$  gives, using (4) and that  $f$  is homogeneous,

$$d = \inf\{dy/y, d\} \leq f(dy) = df(y).$$

Hence, multiplying with  $d^{-1}$  (which exists by  $\varepsilon < f(x_0) = \inf\{x_0/y, d\} \leq d$ ), we get

$$e \leq f(y). \quad (172)$$

Combining (171) and (172) yields  $f(y) = e$ .

Finally, from  $(y, d) \in \text{Supp}_{(X, \mathcal{K})}(f, \tilde{\mathcal{T}}; x_0)$  and  $f(y) = e$  it follows that we have  $2^\circ$ .

The implication  $2^\circ \Rightarrow 3^\circ$  is obvious.

$3^\circ \Rightarrow 1^\circ$ . Assume that the pair  $(y, d) \in (X \setminus \{\inf X\}) \times \mathcal{K}$  satisfies  $3^\circ$ . Then using  $f(y) = e$ , the topicality of  $f$  and Theorem 15a), we obtain

$$s_{y,d}(x) = es_{y,d}(x) = f(y)s_{y,d}(x) \leq f(x), \quad \forall x \in X. \quad (173)$$

On the other hand, by  $3^\circ$  we have  $s_{y,d}(x_0) \geq f(x_0)$ , and hence the equality  $s_{y,d}(x_0) = f(x_0)$ , which, by (173), yields  $(y, d) \in \text{Supp}_{(X, \mathcal{K})}(f, \tilde{\mathcal{T}}; x_0)$ .  $\square$

**Remark 57** We recall that for a set  $X$ , a coupling function  $\pi : X \times X \rightarrow A$ , where  $A = (A, \oplus, \otimes)$  is a conditionally complete lattice ordered group, a function  $f : X \rightarrow \bar{A}$  (where  $\bar{A}$  is the minimal enlargement of  $A$ ) and a point  $x_0 \in X$  with  $f(x_0) \in A$ , the  $\pi$ -subdifferential of  $f$  at  $x_0$  is the subset of  $X$  defined [12, 7] by

$$\partial_\pi f(x_0) := \{y_0 \in X \mid \pi(x, y_0) \otimes \pi(x_0, y_0)^{-1} \otimes f(x_0) \leq f(x), \forall x \in X\}. \quad (174)$$

For  $\pi = \pi_\mu$  of (87) some properties of  $\partial_{\pi_\mu} f(x_0)$ , e.g. that it contains the  $\pi$ -support set of  $f$  at  $x_0$  and some other connections between these sets have been proved in [12, Section 8]. This suggests to attempt to introduce, in our framework of a pair  $(X, \mathcal{K})$  satisfying (A0), (A1') and the coupling function  $\varphi : X \times X \rightarrow \bar{\mathcal{K}}$  of (82), the following concept: We shall call  $\varphi$ -subdifferential of a function  $f : X \rightarrow \bar{\mathcal{K}}$  at a point  $x_0$  with  $f(x_0) \in \mathcal{K}$ , the subset of  $X$  defined by

$$\partial_\varphi f(x_0) := \{y_0 \in X \mid (x/y_0)(x_0/y_0)^{-1} f(x_0) \leq f(x), \forall x \in X\}. \quad (175)$$

The proofs of the properties of  $\partial_{\pi_\mu} f(x_0)$  given in [12, Section 8] lean heavily on the fact that for each  $x \in X$  the partial function  $\pi_\mu(x, \cdot)$  is topical, but in the case of the coupling function  $\varphi : X \times X \rightarrow \bar{\mathcal{K}}$  of (82) for each  $x \in X$  the partial function  $\varphi(x, \cdot)$  is anti-topical (see Lemma 27b)). Therefore those proofs of [12] cannot be adapted to our general framework.

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